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We consider the vacuum energy of the electromagnetic field in the background of spherically symmetric dielectrics, subject to a cut-off frequency in the dispersion relations. The effect of this frequency dependent boundary condition between media is described in terms of the *incomplete* ζ -functions of the problem. The use of the Debye asymptotic expansion for Bessel functions allows to determine the dominant (volume, area, ...) terms in the Casimir energy. The application of these expressions to the case of a gas bubble immersed in water is discussed, and results consistent with Schwinger's proposal about the role the Casimir energy plays in sonoluminescence are found.

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I. INTRODUCTION

The Casimir effect [1–3] arises as a distortion of the vacuum energy of quantized fields due to the presence of boundaries (or nontrivial topologies) in the quantization domain. This effect, which has a quantum nature associated with the zero-point oscillations in the vacuum state, is significant in diverse areas of physics, from statistical physics to elementary particle physics and cosmology.

In particular, in the last years, there has been a great interest in the Casimir energy of the electromagnetic field in the presence of dielectric media, due to Schwinger's suggestion [4] that it could play a role in the explanation of the phenomenon of sonoluminescence [5].

There are essentially two approaches to the subject: One of them consists in summing up retarded van der Waals forces between individual molecules [6,7]. The second one makes use of quantum field theory to evaluate the vacuum energy of the electromagnetic field in the background of dielectric media (see e.g. [8–12,2,13–16]). The relation between these two approaches is not well established. Only for a dilute ball, and up to second order in a perturbative expansion, both methods have been shown to yield the same answer [17,18].

Regarding the relevance of the Casimir effect to sonoluminescence, the results obtained by different groups through several calculation techniques (as Green's functions methods, van der Waals forces, ζ -function methods and asymptotic developments for the density of states - see references [19–24,17,25–30] among others) are rather controversial, and some basic issues remain to be clarified.

In particular, there is no agreement about the renormalization necessary to remove the singularities appearing in the vacuum energy, a fact that renders the physical interpretation of finite parts difficult. But this inconveniences may have their origin in the fact that the models usually employed in describing dielectric media mostly do not incorporate a realistic frequency dependent dispersion relation, then leading to an inadequate ultraviolet behavior.

In a recent paper [31], a nonmagnetic dielectric ball with a frequency dependent permittivity (a high frequency approximation to the Drude model) has been considered. There, it has been shown that a very simple pole structure results for the corresponding ζ -function, and only a volume energy counterterm (to be absorbed in the mass density of the material) is needed to render the Casimir energy finite. Neither surface nor curvature counterterms are needed.

With the ultraviolet behavior under control, it makes sense to analyze finite parts of the Casimir energy for realistic media. In this context, the analysis of the simple model to be considered in this article is a step in the direction of incorporating finite frequency contributions.

It is our aim to contribute to the understanding of the problem by studying a model which incorporates a frequency cut-off Ω in the boundary conditions at the separation between dielectrics, to emulate the behavior of real dielectrics. To this end, we will assume that this boundary is completely transparent for frequencies greater than Ω , for which the dielectric constants take the values corresponding to the vacuum. A similar dispersion relation has been considered in [32] for the case of a dilute medium.

In what follows, we will evaluate the Casimir energy for an arrangement of two such media with spherical symmetry, by adding the eigenfrequencies of the system. The existence of a cut-off will allow us to subtract the contribution of high frequencies, $\omega > \Omega$. On the other hand, low frequency contributions will be represented in terms of the incomplete ζ -functions of the problem, as introduced in [33]. Finally, the asymptotic uniform expansions for Bessel function will allow for the necessary analytic extensions and, for a large cut-off Ω , for the identification of volume and surface contributions to the Casimir energy.

These results will be applied to the case of a gas bubble in water, a situation of interest to sonoluminescence. As we will see, our results seem to support Schwinger's proposal about the role played by Casimir energy in this phenomenon.

The paper is organized as follows. The model is presented in Section II, where the analytic extensions of the incomplete ζ -functions are constructed. The (finite) number of modes effectively contributing for each angular momentum is evaluated in Section III. The bulk and surface contribution to the Casimir energy are obtained in Section IV. In Section V we evaluate the electromagnetic pressure on the separation between dielectrics and the change of vacuum energy with respect to the volume. The application of these results to sonoluminescence is also discussed there. In Section VI we present a summary and discussion of our results. Finally, in Appendix A the representation of the vacuum energy as an integral on the complex plane which we are employing is justified from a mathematical point of view.

II. THE MODEL AND ITS INCOMPLETE ζ -FUNCTION

A. The model

Our aim is to evaluate the Casimir energy of a spherical dielectric bubble of radius a and indices (relative to the vacuum) $\mu_1(\omega), \epsilon_1(\omega)$, immersed in a second medium of indices $\mu_2(\omega), \epsilon_2(\omega)$, as a rough model of real dielectric media for which $\mu_i(\omega), \epsilon_i(\omega)$, $i = 1, 2$, are constants up to a common cut-off frequency Ω , while their values for $\omega > \Omega$ are those of the vacuum. This last restriction reflects itself in a frequency-dependent boundary condition for the electromagnetic field at the separation between the dielectric media, making the boundary completely transparent for those modes of frequency greater than the cut-off Ω .

We will evaluate the Casimir energy of the electromagnetic field in this arrangement of media by summing its eigenfrequencies up to the cut-off, i.e. over all $\omega \leq \Omega$. Notice that disregarding, under the above mentioned conditions, the contribution of modes with frequency $\omega > \Omega$ amounts to redefining of the zero energy level by the subtraction of a (divergent but) a -independent quantity.

Consequently, for $\omega \leq \Omega$ the electromagnetic field satisfies dielectric boundary conditions at the surface of the ball, which in spherical coordinates lead to

$$E_{\theta,\phi}|_{r=a^+} = E_{\theta,\phi}|_{r=a^-}, \quad \frac{1}{\mu_1} B_{\theta,\phi}|_{r=a^+} = \frac{1}{\mu_2} B_{\theta,\phi}|_{r=a^-}. \quad (1)$$

Inside the dielectrics, the electric field satisfies the Helmholtz equation,

$$\Delta \vec{E} + \mu\epsilon \frac{\omega^2}{c^2} \vec{E} = 0, \quad (2)$$

and similarly for the magnetic field, \vec{B} . One can consider the transversal electric (TE) modes, taking the electric field as

$$\vec{E}_{l,m} = f_l(r) \vec{L} Y_{l,m}(\theta, \phi), \quad (3)$$

and separately the transversal magnetic modes (TM), with the magnetic field given by

$$\vec{B}_{l,m} = g_l(r) \vec{L} Y_{l,m}(\theta, \phi), \quad (4)$$

with $l = 1, 2, \dots$ in both cases. In the previous equations

$$\vec{L} = -i \vec{r} \times \vec{\nabla} = -i \hat{\varphi} \partial_\theta + i \hat{\theta} \frac{1}{\sin \theta} \partial_\varphi. \quad (5)$$

For the TE modes, the boundary conditions (1) imply

$$f_l(r)|_{r=a^+} = f_l(r)|_{r=a^-}, \quad \frac{1}{\mu_2} \partial_r [r f_l(r)] \Big|_{r=a^+} = \frac{1}{\mu_1} \partial_r [r f_l(r)] \Big|_{r=a^-}. \quad (6)$$

For the TM modes, the same conditions reduce to

$$\left. \frac{1}{\mu_2} g_l(r) \right|_{r=a^+} = \left. \frac{1}{\mu_1} g_l(r) \right|_{r=a^-}, \quad \left. \frac{1}{\mu_2 \epsilon_2} \partial_r [r g_l(r)] \right|_{r=a^+} = \left. \frac{1}{\mu_1 \epsilon_1} \partial_r [r g_l(r)] \right|_{r=a^-}. \quad (7)$$

Then, for example, we get for $f_l(r)$

$$\frac{1}{r} \frac{d^2}{dr^2} [r f_l(r)] - \frac{l(l+1)}{r^2} f_l(r) = -\mu_{1,2} \epsilon_{1,2} \frac{\omega^2}{c^2} f_l(r), \quad (8)$$

for $r \neq a$ which, together with the boundary conditions in Eq. (6), implies that $f_l(r)$ is a continuous piecewise differentiable function with a discontinuous first derivative at $r = a$.

In order to have a discrete spectrum, we enclose the system inside a large concentric conducting sphere of radius $R \gg a$, obtaining also the Dirichlet condition, $f_l(r) = 0$ at $r = R$, for the functions in the domain of the relevant differential operator. We will take the $R \rightarrow \infty$ limit at the end of the calculation.

In Appendix A, we show that the eigenfrequencies corresponding to TE modes are determined by the zeroes of the function

$$\begin{aligned} \Delta_{l+1/2}^{TE}(z) \equiv & \mathcal{J}_{l+1/2}(\bar{z}_1) \left\{ \mathcal{Y}_{l+1/2}(\bar{z}_0) \mathcal{J}'_{l+1/2}(\bar{z}_2) - \mathcal{J}_{l+1/2}(\bar{z}_0) \mathcal{Y}'_{l+1/2}(\bar{z}_2) \right\} \\ & - \xi \mathcal{J}'_{l+1/2}(\bar{z}_1) \left\{ \mathcal{Y}_{l+1/2}(\bar{z}_0) \mathcal{J}_{l+1/2}(\bar{z}_2) - \mathcal{J}_{l+1/2}(\bar{z}_0) \mathcal{Y}_{l+1/2}(\bar{z}_2) \right\}, \end{aligned} \quad (9)$$

where

$$\mathcal{J}_{l+1/2}(w) = w j_l(w) = \sqrt{\frac{\pi w}{2}} J_{l+1/2}(w), \quad \mathcal{Y}_{l+1/2}(w) = w y_l(w) = \sqrt{\frac{\pi w}{2}} Y_{l+1/2}(w) \quad (10)$$

are the Riccati - Bessel functions. In Eq. (9), $z = a(\omega/c)$, $\bar{z}_{1,2} = z \sqrt{\epsilon_{1,2} \mu_{1,2}}$, $\bar{z}_0 = zR/a \sqrt{\epsilon_2 \mu_2}$, and $\xi = \sqrt{\frac{\epsilon_1 \mu_2}{\epsilon_2 \mu_1}}$. In the same Appendix we show that the zeroes of the function $\Delta_{l+1/2}^{TE}(z)$ in the open right half plane of the variable z are all real and simple.

For the TM modes, the same analysis can be done for the function $\bar{g}_l(r) \equiv \frac{1}{\mu} g_l(r)$, defined for $r \neq a$ and satisfying the Neumann boundary condition, $\bar{g}'_l(r) = 0$ at $r = R$. In this case, the eigenfrequencies are given by the zeroes of the function

$$\begin{aligned} \Delta_{l+1/2}^{TM}(z) = & \mathcal{J}_{l+1/2}(\bar{z}_1) \left\{ \mathcal{Y}'_{l+1/2}(\bar{z}_2) \mathcal{J}'_{l+1/2}(\bar{z}_0) - \mathcal{J}'_{l+1/2}(\bar{z}_2) \mathcal{Y}'_{l+1/2}(\bar{z}_0) \right\} \\ & - \frac{1}{\xi} \mathcal{J}'_{l+1/2}(\bar{z}_1) \left\{ \mathcal{Y}_{l+1/2}(\bar{z}_2) \mathcal{J}'_{l+1/2}(\bar{z}_0) - \mathcal{J}_{l+1/2}(\bar{z}_2) \mathcal{Y}'_{l+1/2}(\bar{z}_0) \right\} \end{aligned} \quad (11)$$

contained in the open right half plane of the variable z , which are also real and simple.

In order to simplify our calculations, in what follows we will consider both media to be nonmagnetic ($\mu_1 = 1 = \mu_2$ for all frequencies), while keeping $\sqrt{\epsilon_1} = n_1$ and $\sqrt{\epsilon_2} = n_2$ arbitrary for $\omega \leq \Omega$.

B. Vacuum energy and incomplete ζ -functions

We will be interested in evaluating differences between vacuum energies corresponding to situations differing in the value of the radius a . Then, as remarked above, we can disregard the contributions of those modes with frequencies $\omega > \Omega$ since, being independent of the position of the boundary (and also of the low frequency refraction indices), they cancel out (whatever the regularization employed in defining the vacuum energy would be). This simply amounts to performing an a -independent subtraction, which is nothing but redefining the zero energy level¹.

Therefore, for the TE modes, we must evaluate the (finite) sum

¹For example, in the framework of the ζ -function regularization, the contribution of the frequencies greater than the cut-off can be defined as the analytic extension to $s = -1$ of a series convergent for $\Re(s)$ large enough:

$$E^{TE}(a) = \sum_{\nu=3/2}^{\nu_0} 2\nu \sum_{n=1}^{N_\nu} \frac{1}{2} \hbar \omega_{\nu,n} = \frac{\hbar c}{a} \sum_{\nu=3/2}^{\nu_0} \nu \sum_{n=1}^{N_\nu} z_{\nu,n}, \quad (13)$$

and a similar expression for the TM modes. In Eq. (13), N_ν is the number of positive zeroes of $\Delta_\nu^{TE}(z)$, $z_{\nu,n}$, which are less than or equal to $x = a\Omega/c$. The degeneracy due to the spherical symmetry is $2\nu = 2l + 1$, and ν_0 is the maximum value of ν for which $N_\nu \geq 1$.

We are interested in an analytic (rather than numeric) evaluation of Eq. (13). So, although this is a finite sum, we will employ the summation method developed in [33], based on the evaluation of an *incomplete* ζ -function. We can use the following representation:

$$\sum_{n=1}^{N_\nu} z_{\nu,n} = \sum_{n=1}^{N_\nu} z_{\nu,n}^{-s} \Big|_{s=-1}, \quad (14)$$

where the sum in the right hand side exists as an analytic function² of $s \in \mathbf{C}$.

Since $\Delta_\nu^{TE}(z)$ has only real zeros, and its non-vanishing zeros are all simple (see Appendix A), we can employ the Cauchy theorem to represent the sum in the r.h.s. of (14) as an integral on the complex plane,

$$\sum_{n=1}^{N_\nu} z_{\nu,n}^{-s} = \frac{1}{2\pi i} \oint_C z^{-s} \frac{\Delta_\nu^{TE'}(z)}{\Delta_\nu^{TE}(z)} dz, \quad (15)$$

where the curve C encircles the first N_ν positive zeros of $\Delta_\nu^{TE}(z)$ counterclockwise.

For $\Re(s)$ large enough, the contour C can be deformed into two straight vertical lines, one crossing the horizontal axis at $\Re(z) = x$ and the other at $\Re(z) = 0^+$. Indeed, the integrand can be expressed in terms of modified Bessel functions through the substitutions [34]

$$\begin{aligned} J_\nu(e^{i\frac{\pi}{2}} w) &= e^{i\frac{\pi}{2}\nu} I_\nu(w), & Y_\nu(e^{i\frac{\pi}{2}} w) &= e^{i\frac{\pi}{2}(\nu+1)} I_\nu(w) - \frac{2}{\pi} e^{-i\frac{\pi}{2}\nu} K_\nu(w), \\ J'_\nu(e^{i\frac{\pi}{2}} w) &= e^{i\frac{\pi}{2}(\nu-1)} I'_\nu(w), & Y'_\nu(e^{i\frac{\pi}{2}} w) &= e^{i\frac{\pi}{2}\nu} I'_\nu(w) - \frac{2}{\pi} e^{-i\frac{\pi}{2}(\nu+1)} K'_\nu(w), \end{aligned} \quad (16)$$

relations valid for $-\pi < \arg(w) \leq \pi/2$. Thus, we get (see Eq. (9))

$$\begin{aligned} \Delta_\nu^{TE}(x + iy) &= -e^{\frac{i\pi}{2}(\nu+1/2)} \sqrt{\frac{n_1 \pi R}{2a}} v^{\frac{3}{2}} \left\{ K_\nu\left(\frac{n_2 R v}{a}\right) [-n_1 I_\nu(n_2 v) I'_\nu(n_1 v) \right. \\ &\quad \left. + n_2 I_\nu(n_1 v) I'_\nu(n_2 v)] + I_\nu\left(\frac{n_2 R v}{a}\right) [n_1 K_\nu(n_2 v) I'_\nu(n_1 v) - n_2 I_\nu(n_1 v) K'_\nu(n_2 v)] \right\}, \end{aligned} \quad (17)$$

where $v = (-ix + y)$. Taking into account the asymptotic behavior of the modified Bessel functions for large arguments [34], it is easily seen that, for $0 < x \neq z_{\nu,n}, \forall n$, the integral

$$\zeta_\nu^{TE}(s, x) \equiv \frac{-1}{2\pi i} \int_{x-i\infty}^{x+i\infty} z^{-s} \frac{\Delta_\nu^{TE'}(z)}{\Delta_\nu^{TE}(z)} dz, \quad (18)$$

$$E_0(s) = \frac{1}{2} \hbar \Omega \sum_{\nu} 2\nu \sum_{\omega_{\nu,n}^{(0)} > \Omega} \left(\frac{\omega_{\nu,n}^{(0)}}{\Omega} \right)^{-s}, \quad (12)$$

where $\nu = l + 1/2$, and $\omega_{\nu,n}^{(0)}$ are the zeroes of $\Delta_\nu^{TE}(a\omega/c)$ and $\Delta_\nu^{TM}(a\omega/c)$ taken with $n_1 = 1 = n_2$, i.e. in a situation indistinguishable from one where the external sphere contains only vacuum. Obviously, $E_0(s)$ is independent of a , n_1 , and n_2 . It can only depend on R and Ω .

²Notice that the sum in the r.h.s. of Eq. (14) evaluated at $s = 0$ gives N_ν , the number of eigenfrequencies contributing effectively to the Casimir energy of the field for a given value of the angular momentum $l = \nu - 1/2$, once the a -independent subtraction adopted to define it has been made.

converges absolutely and uniformly to an analytic function in the open half-plane $\Re(s) > 1$. Without loss of generality, and for calculational convenience, we will restrict ourselves to real values of s , and evaluate the function in Eq. (18) on the half-line $s > 1$, from which it can be meromorphically extended to the whole complex s -plane.

Therefore, for $s > 1$,

$$\sum_{n=1}^{N_\nu} z_{\nu,n}^{-s} = \zeta_\nu^{TE}(s, 0^+) - \zeta_\nu^{TE}(s, x). \quad (19)$$

Moreover, since the left hand side of (19) is holomorphic in s , the singularities of $\zeta_\nu^{TE}(s, x)$ must be independent of x . In particular, this allows us to write the vacuum energy as the analytic extension

$$E^{TE}(a) = \frac{\hbar c}{a} \sum_{\nu=3/2}^{\nu_0} \nu \left[\zeta_\nu^{TE}(s, 0^+) - \zeta_\nu^{TE}(s, x) \right] \Bigg|_{s \rightarrow -1}. \quad (20)$$

Entirely similar conclusions are obtained for the TM case.

Moreover, for $\Re(w) > 0$ we have [34]

$$\begin{aligned} I_\nu(e^{-\imath\pi} w) &= e^{-\imath\pi\nu} (I_\nu(w^*))^*, & K_\nu(e^{-\imath\pi} w) &= e^{\imath\pi\nu} (K_\nu(w^*))^* + \imath\pi (I_\nu(w^*))^*, \\ I'_\nu(e^{-\imath\pi} w) &= e^{-\imath\pi(\nu-1)} (I'_\nu(w^*))^*, & K'_\nu(e^{-\imath\pi} w) &= e^{\imath\pi(\nu+1)} (K'_\nu(w^*))^* + \imath\pi e^{\imath\pi} (I'_\nu(w^*))^*. \end{aligned} \quad (21)$$

So, changing the integration variable in Eq. (18) by $z \rightarrow (y - \imath)x$, and calling $t = (y - \imath)x/\nu$, we can straightforwardly write for real $s > 1$

$$\zeta_\nu^{TE}(s, x) = \Re \left\{ \frac{-\nu^{-s}}{\pi} e^{-i\frac{\pi}{2}(s+1)} \int_{-iz}^{\infty-iz} t^{-s} \frac{d(\ln \Delta_\nu^{TE}(\imath\nu t))}{dt} dt \right\}, \quad (22)$$

where we have now called $z = x/\nu > 0$.

A similar expression is obtained for $\zeta_\nu^{TM}(s, x)$, corresponding to the TM modes.

C. The analytic extension of incomplete ζ -functions

In order to construct the analytic extension of $\zeta_\nu^{TE}(s, x)$ to $s \simeq -1$, we will subtract and add to the integrand in (22) the first few terms obtained from the uniform asymptotic (Debye) expansion [34] of the Bessel functions appearing in the expression of $\Delta_\nu^{TE}(\imath\nu t)$ (see Eq. (17)), which is valid for large ν with fixed t :

$$\frac{d \ln \Delta_\nu^{TE}(\imath\nu t)}{dt} = D_\nu^{TE}(t) + \mathcal{O}(\nu^{-2}), \quad (23)$$

where

$$D_\nu^{TE}(t) = \nu D_{TE}^{(1)}(t) + D_{TE}^{(0)}(t) + \nu^{-1} D_{TE}^{(-1)}(t). \quad (24)$$

In the above expression, the functions $D_{TE}^{(k)}(t)$, $k = 1, 0, -1$, explicitly shown in Appendix B, are algebraic functions of t . Notice that we have discarded contributions coming from terms containing $K_\nu(\frac{n_2 R \nu t}{a})$ or its derivative (see Eq. (17)), since they vanish exponentially when $R \rightarrow \infty$. We will see that this approximation allows for the identification of the volume, surface, ... contributions to the vacuum energy.

So, we must consider the integral

$$\int_{-iz}^{\infty-iz} t^{-s} \frac{d(\ln \Delta_\nu^{TE}(\imath\nu t))}{dt} dt = \int_{-iz}^{\infty-iz} t^{-s} D_\nu^{TE}(t) dt ++ \int_{-iz}^{\infty-iz} t^{-s} \left\{ \frac{d(\ln \Delta_\nu^{TE}(\imath\nu t))}{dt} - D_\nu^{TE}(t) \right\} dt. \quad (25)$$

The second integral in the right hand side of Eq. (25) converges for $s > -2$, since the integrand can be estimated by means of the contribution of the next ($\mathcal{O}(\nu^{-2})$) term in the Debye expansion (see Eq. (23)), which behaves as $\mathcal{O}(t^{-s-3})$ for large $|t|$. This term could be evaluated numerically at $s = -1$. This will not be done in this paper.

The Debye expansion applied to the TM modes case gives

$$D_{\nu}^{TM}(t) = \nu D_{TM}^{(1)}(t) + D_{TM}^{(0)}(t) + \nu^{-1} D_{TM}^{(-1)}(t), \quad (26)$$

instead of Eq. (24), and leads to a decomposition similar to Eq. (25). The algebraic functions $D_{TM}^{(k)}(t)$ are also shown in the Appendix B.

In what follows, we will evaluate only the first integral in the right hand side of (25) (and the analogous expression for the TM modes), retaining only those terms of its expansion in powers of ν^{-1} which are consistent with the approximation made in Eq. (23).

Notice that the integrand, $D_{\nu}^{TE}(t)$ ($D_{\nu}^{TM}(t)$ for TM modes), is an algebraic function behaving as $\mathcal{O}(t^0)$ for large $|t|$ (see Appendix B). So, the integral converges absolutely and uniformly for $s > 1$, where it defines an analytic function which can be meromorphically extended to the region of interest of the parameter s . As we will see, this extension reveals the singularities of $\zeta_{\nu}^{TE}(s, x)$ ($\zeta_{\nu}^{TM}(s, x)$) as simple poles with x -independent residues (a necessary condition to get a finite result for any s in Eq. (19)). Notice that this statement must be valid for the contribution of each order in ν .

We begin the calculation by considering the terms of dominant order in the Debye expansion³, $\nu D_{TE}^{(1)}(t)$. By virtue of the analyticity of the integrand (see eqs. (B1)), for $s > 1$ we can deform the path of integration to write

$$\begin{aligned} \int_{-iz}^{\infty - iz} t^{-s} \nu D_{TE}^{(1)}(t) dt &= \nu \int_{-iz}^1 t^{-s-1} \left(\sqrt{1+n_1^2 t^2} - \sqrt{1+n_2^2 t^2} + \sqrt{1 + \frac{n_2^2 R^2 t^2}{a^2}} \right) dt + \\ \nu \int_1^{\infty} t^{-s} \left\{ \frac{1}{t} \left(\sqrt{1+n_1^2 t^2} - \sqrt{1+n_2^2 t^2} + \sqrt{1 + \frac{n_2^2 R^2 t^2}{a^2}} \right) - \left(n_1 - n_2 + \frac{n_2 R}{a} + \frac{\frac{1}{n_1} - \frac{1}{n_2} + \frac{a}{n_2 R}}{2 t^2} \right) \right\} dt \quad (27) \\ &+ \nu \int_1^{\infty} t^{-s} \left(n_1 - n_2 + \frac{n_2 R}{a} + \frac{\frac{1}{n_1} - \frac{1}{n_2} + \frac{a}{n_2 R}}{2 t^2} \right) dt. \end{aligned}$$

The first integral in the r.h.s. of eq. (27), which contains the whole dependence on $x = \nu z$, is holomorphic in s and can be directly evaluated at the required value of this parameter. On the half-line $(1, \infty)$ we have subtracted and added the first terms in the series expansion of $D_{TE}^{(1)}(t)$ for large t , which makes the second integral convergent for $s > -2$. The third one must be evaluated for $s > 1$ and then analytically continued to the relevant values of s . This can be exactly done, its contribution to $\zeta_{\nu}^{TE}(s, x)$ in Eq. (22) being

$$\Delta_1 \zeta_{\nu}^{TE}(s, x)|_{\text{Sing.}} = \frac{\nu^{1-s} \cos\left(\frac{\pi}{2}(1+s)\right)}{2 a n_1 n_2 \pi R} \left(\frac{2 n_1 n_2 R (a n_1 - a n_2 + n_2 R)}{1-s} + \frac{-(a^2 n_1) + a n_1 R - a n_2 R}{1+s} \right). \quad (28)$$

This expression is analytic at $s = 0$ and has simple poles at $s = \pm 1$, which are the only singularities of the dominant contribution to $\zeta_{\nu}^{TE}(s, x)$ in this asymptotic expansion for $\Re(s) > -2$. In particular, its residue at $s = -1$ is

$$\text{Res } \Delta_1 \zeta_{\nu}^{TE}(s, x)|_{s=-1} = - \frac{\nu^2 (a n_1 + (n_2 - n_1) R)}{2 n_1 n_2 \pi R}. \quad (29)$$

Notice that, as anticipated, up to this order in the Debye expansion the residue is independent of x .

For example, for the dominant contribution to $\zeta_{\nu}^{TE}(s, x)$ (which coincides up to this order with $\zeta_{\nu}^{TM}(s, x)$ - see footnote 3),

$$\zeta_{\nu}^{TE}(s, x) = \Delta_1 \zeta_{\nu}^{TE}(s, x) (1 + \mathcal{O}(\nu^{-1})), \quad (30)$$

one straightforwardly obtains the Laurent expansion around $s = -1$

³ Since $D_{TM}^{(1)}(t)$ is identical to $D_{TE}^{(1)}(t)$ (see eqs. (B1) and (B5)), we obtain the same result for the dominant contribution to $\zeta_{\nu}^{TM}(s, x)$.

$$\begin{aligned}
\Delta_1 \zeta_\nu^{TE}(s, x) &= \Delta_1 \zeta_\nu^{TM}(s, x) = -\frac{\nu^2 (a n_1 + (n_2 - n_1) R)}{2 n_1 n_2 \pi R (1 + s)} \\
&+ \Re \left\{ \frac{-\nu^2}{4 n_1 n_2 \pi R} (a n_1 + R (n_2 - n_1) (1 - 2 n_1 n_2) + 2 n_1 n_2 R \left(n_2 R + i a z \left(\sqrt{1 + e^{-i \pi} n_1^2 z^2} \right. \right. \right. \\
&\quad \left. \left. \left. - \sqrt{1 + e^{-i \pi} n_2^2 z^2} + \sqrt{1 + \frac{e^{-i \pi} n_2^2 R^2 z^2}{a^2}} \right) \right) \right) + 2 (a n_1 \log(n_2 R) - n_1 R \log(n_2) \\
&\quad + n_2 R \log(n_1)) \log(2) + (n_2 - n_1) (2 n_1 n_2) R - 2 n_1 n_2^2 \frac{R^2}{a} - 2 (a n_1 + (-n_1 + n_2) R) \log(\nu) \\
&\quad - 2 n_2 R \log \left(-i n_1 z + \sqrt{1 + e^{-i \pi} n_1^2 z^2} \right) + 2 n_1 R \log \left(-i n_2 z + \sqrt{1 + e^{-i \pi} n_2^2 z^2} \right) \\
&\quad \left. \left. - 2 a n_1 \log \left(-i n_2 R z + a \sqrt{1 + \frac{e^{-i \pi} n_2^2 R^2 z^2}{a^2}} \right) \right) \right\} + \mathcal{O}(1 + s).
\end{aligned} \tag{31}$$

On the other hand, for $x \rightarrow 0^+$ a similar calculation leads to

$$\begin{aligned}
\Delta_1 \zeta_\nu^{TE}(s, x = 0^+) &= \Delta_1 \zeta_\nu^{TM}(s, x = 0^+) = -\frac{\nu^2 (a n_1 + (n_2 - n_1) R)}{2 n_1 n_2 \pi R (1 + s)} \\
&+ \frac{\nu^2}{4 n_1 n_2 \pi} (2 n_1 \log(2 n_2) - 2 n_2 \log(2 n_1) + (n_1 - n_2) (1 - 2 \log(\nu))) + \mathcal{O}(s + 1).
\end{aligned} \tag{32}$$

Similar calculations are required in order to get the contributions to $\zeta_\nu^{TE}(x, s)$ ($\zeta_\nu^{TM}(x, s)$) coming from the next to leading terms in Eq. (24) (Eq. (26) respectively). In fact, from Eqs. (B2) (Eq. (B6) for the TM case) it is easily seen that $D_{TE}^{(0)}(t)$ ($D_{TM}^{(0)}(t)$) $\sim t^{-3}$. So, its contributions to the first integral in the right hand side of Eq. (25) (or the equivalent for TM) converges to an analytic function for $s > -2$, and does not affect the residue of $\zeta_\nu^{TE}(x, s)$ ($\zeta_\nu^{TM}(x, s)$) at $s = -1$.

For the second order term in the Debye expansion of the difference $[\zeta_\nu^{TE}(s, x = 0^+) - \zeta_\nu^{TE}(s, x)]$ around $s = -1$, a straightforward calculation leads to

$$\begin{aligned}
\Delta_0 \zeta_\nu^{TE}(s, 0^+) - \Delta_0 \zeta_\nu^{TE}(s, x) &= \nu \left\{ -\frac{1}{4 n_1} \Theta(n_1 x - \nu) - \frac{1}{4 n_2} \Theta(n_2 x - \nu) - \frac{a}{4 n_2 R} \Theta(n_2 R x / a - \nu) \right. \\
&\quad \left. + \frac{2}{\pi (n_> + n_<)} \Theta(\nu_> - \nu) F \left(\arcsin \sqrt{\frac{(n_> + n_<)(\nu_> - u)}{(n_> - n_<)(\nu_> + u)}}, \frac{n_> - n_<}{n_> + n_<} \right) \right\} + \mathcal{O}(s + 1).
\end{aligned} \tag{33}$$

Here, $\Theta(w)$ is a step function (vanishing for $w < 0$ and equal to 1 for $w > 0$), $F(\varphi, k)$ is the elliptic integral of the first kind, $n_<$ ($n_>$) is the min(max) $\{n_1, n_2\}$, and $u = \max\{\nu, \nu_<\}$, with $\nu_< = n_< x$ and $\nu_> = n_> x$.

Similarly, for the TM modes we get

$$\begin{aligned}
\Delta_0 \zeta_\nu^{TM}(s, 0^+) - \Delta_0 \zeta_\nu^{TM}(s, x) &= \nu \left\{ -\frac{1}{4 n_1} \Theta(n_1 x - \nu) - \frac{1}{4 n_2} \Theta(n_2 x - \nu) + \frac{a}{4 n_2 R} \Theta(n_2 R x / a - \nu) \right. \\
&\quad \left. + \frac{n_<^2}{\pi n_>^3} \Theta(n_> x - \nu) \Pi \left(\arcsin \sqrt{\frac{\nu_>^2 - u^2}{\nu_>^2 - \nu_<^2}}, 1 - \frac{n_<^4}{n_>^4}, \sqrt{1 - \frac{n_<^2}{n_>^2}} \right) \right\} + \mathcal{O}(s + 1),
\end{aligned} \tag{34}$$

where $\Pi(\varphi, n, k)$ is the elliptic integral of the third kind and $u = \max\{\nu, \nu_<\}$.

Finally, notice that $D_{TE}^{(-1)}(t)$ and $D_{TM}^{(-1)}(t) \approx t^{-2}$ (see Eqs. (B3) and (B7)). So, they do contribute to the poles at $s = -1$. We get

$$\Delta_{-1} \zeta_\nu^{TE}(s, x) = \Delta_{-1} \zeta_\nu^{TM}(s, x) = \frac{n_1 - n_2}{8 n_1 n_2 \pi (1 + s)} - \frac{a}{8 n_2 \pi R (1 + s)} + \mathcal{O}(s + 1)^0, \tag{35}$$

although the finite parts (which will be not given explicitly here) are different for TE and TM. Notice that the residues are independent of x , and cancel out when the difference in Eq. (19) (or the equivalent for the TM case) is taken.

In the following Section we will evaluate N_ν (i.e., the number of modes contributing in Eq. (19)) as a function of ν , and in Section IV we will get their contributions to the vacuum energy.

III. THE NUMBER OF CONTRIBUTING MODES

In this Section we address ourselves to the determination of ν_0 in Eq. (13), i.e., the maximum value of ν for which $N_\nu \geq 1$. As before, we will follow the method established in [33].

First, notice that

$$N_\nu(x) \equiv \sum_{n=1}^{N_\nu} z_{\nu,n}^{-s} \Big|_{s=0} = [\zeta_\nu^{TE}(s, 0^+) - \zeta_\nu^{TE}(s, x)] \Big|_{s=0} \quad (36)$$

is a step function of x , having a discontinuity of height 1 at each positive zero $z_{\nu,n}$ of the function $\Delta_\nu^{TE}(z)$ in Eq. (9). Then, $\nu_0(x)$ can be determined from the condition

$$N_{\nu_0}(x) = N_{\nu_0}(z_{\nu_0,1} + 0) = 1, \quad (37)$$

with $N_{\nu_0}(z_{\nu_0,1} - 0) = 0$.

For the dominant order of $D_\nu^{TE}(t)$, Eq. (B1), taking into account Eq. (28) and the fact that the second integral in the r.h.s. of Eq. (27) is (finite and) real at $s = 0$, it is straightforward to obtain, from Eqs. (22) and (27), that

$$\begin{aligned} \Delta_1 \zeta_\nu^{TE}(s=0, x) = & -\frac{\nu}{2} - \Re \left\{ \frac{i\nu}{\pi} \left(\sqrt{1 + e^{-i\pi} n_1^2 z^2} - \sqrt{1 + e^{-i\pi} n_2^2 z^2} + \sqrt{1 + e^{-i\pi} \frac{n_2^2 R^2 z^2}{a^2}} \right. \right. \\ & \left. \left. - \log(1 + \sqrt{1 + e^{-i\pi} n_1^2 z^2}) + \log(1 + \sqrt{1 + e^{-i\pi} n_2^2 z^2}) - \log(1 + \sqrt{1 + e^{-i\pi} \frac{n_2^2 R^2 z^2}{a^2}}) \right) \right\}. \end{aligned} \quad (38)$$

In particular, for $x \rightarrow 0^+$,

$$\Delta_1 \zeta_\nu^{TE}(s=0, x=0^+) = -\frac{\nu}{2} \quad (39)$$

Similarly, from Eq. (B2), a straightforward calculation leads to

$$\begin{aligned} & \Delta_0 \zeta_\nu^{TE}(s=0, x=0^+) - \Delta_0 \zeta_\nu^{TE}(s=0, x) = \\ & = \Re \left\{ \frac{-i}{4\pi} [\log(1 + e^{-i\pi} n_1^2 z^2) + \log(1 + e^{-i\pi} n_2^2 z^2) + \log(a^2 + e^{-i\pi} n_2^2 R^2 z^2)] \right\}. \end{aligned} \quad (40)$$

Now, calling

$$\tilde{N}_\nu^{TE}(x) \equiv \Delta_1 \zeta_\nu^{TE}(s=0, 0^+) - \Delta_1 \zeta_\nu^{TE}(s=0, x) + \Delta_0 \zeta_\nu^{TE}(s=0, 0^+) - \Delta_0 \zeta_\nu^{TE}(s=0, x) = N_\nu(x) + \mathcal{O}(\nu^{-1}), \quad (41)$$

we see that it gives a *smooth* approximation to the step function in (36) for $\nu \gg 1$. So, following [33], we will approximate $\nu_0(x)$ by the value of ν for which $\tilde{N}_\nu^{TE}(x) = 1/2$.

Since $z = x/\nu$ (with $x = a\Omega/c$), it can be easily seen that $\tilde{N}_\nu^{TE}(x) = 0$ for $\nu > n_2 R x / a$ ($\Rightarrow 1 > n_2 R z / a > n_{1,2} z$), while for $n_2 R x / a > \nu > n_{1,2} x$ we have

$$\tilde{N}_\nu^{TE}(x) = \frac{\nu}{\pi} \left[\sqrt{\frac{n_2^2 R^2 z^2}{a^2} - 1} - \arctan \left(\sqrt{\frac{n_2^2 R^2 z^2}{a^2} - 1} \right) \right] - \frac{1}{4}. \quad (42)$$

Now, we will assume that $\tilde{N}_{\nu_0}^{TE}(x) = 1/2$ for $\nu_0 \lesssim n_2 R x / a$, write $\varepsilon^2 = \frac{n_2^2 R^2 x^2}{\nu_0^2 a^2} - 1$, and determine ε iteratively from the series expansion of the right hand side of Eq. (42) around $\varepsilon = 0$. This leads to

$$\nu_0^{TE} \simeq \frac{n_2 R \Omega}{c} \left\{ 1 - k_1^{TE} \left(\frac{n_2 R \Omega}{c} \right)^{-2/3} + k_2^{TE} \left(\frac{n_2 R \Omega}{c} \right)^{-4/3} + \mathcal{O} \left(\frac{R \Omega}{c} \right)^{-2} \right\}, \quad (43)$$

where

$$k_1^{TE} = 3^{1/3} \frac{3 \pi^{2/3}}{4 \cdot 2^{1/3}}, \quad k_2^{TE} = 3^{2/3} \frac{3 \pi^{4/3}}{160 \cdot 2^{2/3}}. \quad (44)$$

Notice that this result does not depend on a or n_1 , i.e., the radius of the bubble and its refraction index.

For the TM modes, a similar calculation shows that $\tilde{N}_\nu^{TM}(x) = \tilde{N}_\nu^{TE}(x) + 1/2$, and

$$\nu_0^{TM} \simeq \frac{n_2 R \Omega}{c} \left\{ 1 - k_1^{TM} \left(\frac{n_2 R \Omega}{c} \right)^{-2/3} + k_2^{TM} \left(\frac{n_2 R \Omega}{c} \right)^{-4/3} + \mathcal{O} \left(\frac{R \Omega}{c} \right)^{-2} \right\}, \quad (45)$$

with

$$k^{TM} = 3^{-1/3} \frac{3 \pi^{2/3}}{4 \cdot 2^{1/3}}, \quad k_2^{TM} = 3^{-2/3} \frac{3 \pi^{4/3}}{160 \cdot 2^{2/3}}. \quad (46)$$

ν_0^{TM} is also independent of a and n_1 .

IV. THE DOMINANT CONTRIBUTIONS TO THE VACUUM ENERGY

In this Section we will evaluate the contribution to the vacuum energy due to the dominant orders in the Debye expansion of incomplete ζ -functions, obtained in Section II.

A. Bulk contributions

According to the results in Section II, we need the Laurent expansion of $\Delta_1 \zeta_\nu^{TE}(s, x)$ around $s = -1$ (given in Eqs. (31) for arbitrary x , and in Eq. (32) for $x = 0$). As already remarked, the contribution of singular parts to the right hand side of Eq. (19) cancel out, since the residues are independent of x (see Eq. (29)). For the difference of the finite parts for $\nu \leq \nu_0^{TE}$ we get

$$\begin{aligned} & [\Delta_1 \zeta_\nu^{TE}(s, 0^+) - \Delta_1 \zeta_\nu^{TE}(s, x)]|_{s=-1} = \\ & = \nu^2 z \{ \Theta(n_1 x - \nu) G(n_1 z) - \Theta(n_2 x - \nu) G(n_2 z) + \Theta(\nu_0^{TE} - \nu) G(n_2 R z / a) \}, \end{aligned} \quad (47)$$

where

$$G(w) = \frac{\sqrt{w^2 - 1}}{2 \pi} - \frac{\log(w + \sqrt{w^2 - 1})}{2 \pi w}. \quad (48)$$

Replacing $z = x/\nu$ and $x = a\Omega/c$, we obtain for the dominant order contribution to the sum in Eq. (20),

$$\frac{1}{\hbar \Omega} \Delta_1 E^{TE}(a) = \sum_{\nu \leq n_1 a \Omega / c} \nu^2 G\left(\frac{n_1 a \Omega}{\nu c}\right) - \sum_{\nu \leq n_2 a \Omega / c} \nu^2 G\left(\frac{n_2 a \Omega}{\nu c}\right) + \sum_{\nu \leq \nu_0^{TE}} \nu^2 G\left(\frac{n_2 R \Omega}{\nu c}\right), \quad (49)$$

where $\nu = l + 1/2$, with $l = 1, 2, \dots$. Since $\mu^2 G(1/\mu)$ has a pronounced maximum at $\mu \approx 1/2$, the approximation to $E^{TE}(a)$ given by $\Delta_1 E^{TE}(a)$ is justified as long as $a\Omega/c \gg 1$.

Notice that the last term in the right hand side of Eq. (49), the only piece which is a function of R , depends neither on a (the radius of the bubble) nor on n_1 (the refraction index of the internal medium). In fact, as shown in the previous section, ν_0^{TE} is independent of these parameters (see Eq. (43)). Therefore, this is the only piece remaining in the limit $a \rightarrow 0$ (no internal bubble), where the other two terms are vanishing.

In order to sum up the first and second terms in the right hand side of Eq. (49) notice that, even though $f_k(\nu) \equiv \nu^2 G(n_k a \Omega / \nu c)$, $k = 1$ or 2 , and its first derivative are finite and bounded for $\nu \in [\frac{3}{2}, \nu_k]$ (with $\nu_k = [n_k a \Omega / c - 1/2] + 1/2$, where the square bracket denotes the integer part), the second derivative is unbounded near ν_k . This is so because

$$f_k(\nu) = g_k(\nu) + \mathcal{O}(\nu_k - \nu)^{\frac{5}{2}}, \quad (50)$$

where

$$g_k(\nu) = \frac{2\sqrt{2\nu_k}}{3\pi} (\nu_k - \nu)^{\frac{3}{2}}. \quad (51)$$

Therefore, we can subtract and add $g_k(\nu)$ to $f_k(\nu)$, and apply the Euler - Maclaurin summation formula [35] to the difference,

$$\begin{aligned} \sum_{\nu=3/2}^{\nu_k} [f_k(\nu) - g_k(\nu)] &= \int_{3/2}^{\nu_k} [f_k(x) - g_k(x)] dx + \frac{1}{2} ([f_k(3/2) - g_k(3/2)] + [f_k(\nu_k) - g_k(\nu_k)]) + \\ &+ \frac{1}{2} \sum_{\nu=5/2}^{\nu_k} \int_{\nu-1}^{\nu} (x - [x]) (1 - x + [x]) [f_k^{(2)}(x) - g_k^{(2)}(x)] dx. \end{aligned} \quad (52)$$

Since the second derivative in the argument of the last integral is non-positive, it is easy to see that the remainder (this last term) is $\mathcal{O}(n_k a \Omega / c)$. Then, a straightforward calculation leads to

$$\sum_{\nu=3/2}^{\nu_k} [f_k(\nu) - g_k(\nu)] = \frac{(5 - 16\sqrt{2})}{60\pi} \left(\frac{n_k a \Omega}{c} \right)^3 + \frac{2^{3/2}}{3\pi} \left(\frac{n_k a \Omega}{c} \right)^2 + \mathcal{O} \left(\frac{n_k a \Omega}{c} \right). \quad (53)$$

On the other hand,

$$\begin{aligned} \sum_{\nu=3/2}^{\nu_k} g_k(\nu) &= \frac{2^{3/2}}{3\pi} \sqrt{\frac{n_k a \Omega}{c}} \left\{ \zeta \left(\frac{-3}{2}, \alpha_k \right) - \zeta \left(\frac{-3}{2}, \left(\frac{n_k a \Omega}{c} - \frac{1}{2} \right) \right) \right\} = \\ &= \frac{2^{3/2}}{3\pi} \left\{ \frac{2}{5} \left(\frac{n_k a \Omega}{c} \right)^3 - \left(\frac{n_k a \Omega}{c} \right)^2 + \mathcal{O} \left(\frac{n_k a \Omega}{c} \right) \right\}, \end{aligned} \quad (54)$$

where $\alpha_k = (n_k a \Omega / c - 1/2) - [n_k a \Omega / c - 1/2] \in [0, 1)$, and the asymptotic expansion of the Hurwitz ζ -function, $\zeta(s, v)$, for large v [35] has been used in the last step.

Finally, adding the results in Eqs. (53) and (54) (notice that the surface contributions cancel out), taking the difference for $k = 1, 2$, and adding a similar expression coming from the third term in the right hand side of Eq. (49), we get

$$\begin{aligned} \frac{1}{\hbar \Omega} \Delta_1 E^{TE}(a) &= \frac{(n_1^3 - n_2^3)}{12\pi} \left(\frac{a \Omega}{c} \right)^3 + \mathcal{O} \left(\frac{a \Omega}{c} \right) + \\ &+ \frac{1}{12\pi} \left(\frac{n_2 R \Omega}{c} \right)^3 \left\{ 1 - 3 k_1^{TE} \left(\frac{n_2 R \Omega}{c} \right)^{-2/3} + 3 (k_1^{TE^2} + k_2^{TE}) \left(\frac{n_2 R \Omega}{c} \right)^{-4/3} \right\} + \mathcal{O} \left(\frac{R \Omega}{c} \right). \end{aligned} \quad (55)$$

The same result, with $k_{1,2}^{TE} \rightarrow k_{1,2}^{TM}$, is found for $\Delta_1 E^{TM}(a)$.

Eq. (55) shows that the dominant contributions to the vacuum energy in this asymptotic expansion are volume terms, in agreement with [4,23].

There is a term proportional to the volume of the accessible space ($\sim R^3$), with corrections depending on fractional powers of R induced by the cut-off imposed (see Eqs. (43) and (45)). These corrections are independent of n_1 and a . There is also a bulk contribution proportional to the volume of the bubble ($\sim a^3$), which is twice the one obtained for the scalar field case discussed in [33], multiplied by $(n_1^3 - n_2^3)$. So, it is the sign of the difference $(n_1^3 - n_2^3)$ which determines the vacuum energy behavior with respect to the radius of the bubble. In particular, it vanishes for $n_1 = n_2$. This will be further discussed in Section V.

In order to incorporate the first finite-size correction to the vacuum energy, we need the Laurent expansions of the next to leading order in the asymptotic expansions of the ζ -functions around $s = -1$, quoted in Eqs. (33) and (34). For the TE case we get

$$\begin{aligned} \frac{1}{\hbar \Omega} \Delta_0 E^{TE}(a) &= \left(\frac{a\Omega}{c} \right)^{-1} \sum_{\nu=3/2}^{\nu_0^{TE}} \nu \left[\Delta_0 \zeta_\nu^{TE}(s, 0^+) - \Delta_0 \zeta_\nu^{TE}(s, x) \right] \Big|_{s \rightarrow -1} = \\ &= \left(\frac{a\Omega}{c} \right)^{-1} \left\{ -\frac{1}{4n_<} \sum_{\nu \leq \nu_<} \nu^2 - \frac{1}{4n_>} \sum_{\nu \leq \nu_>} \nu^2 - \frac{a}{4n_2 R} \sum_{\nu \leq \nu_0^{TE}} \nu^2 + \right. \\ &\quad \left. + \frac{2}{\pi(n_> + n_<)} \left[K \left(\frac{n_> - n_<}{n_> + n_<} \right) \sum_{\nu \leq \nu_<} \nu^2 + \sum_{\nu_< < \nu \leq \nu_>} \nu^2 F \left(\arcsin \sqrt{\frac{(n_> + n_<)(\nu_> - \nu)}{(n_> - n_<)(\nu_> + \nu)}}, \frac{n_> - n_<}{n_> + n_<} \right) \right] \right\}, \end{aligned} \quad (56)$$

where $\nu = l + 1/2$, with $l = 1, 2, \dots$, $\nu_< = n_< x$, $\nu_> = n_> x$ and $K(k) = F(\pi/2, k)$ is the complete elliptic integral.

Similarly, for the TM modes we have

$$\begin{aligned} \frac{1}{\hbar \Omega} \Delta_0 E^{TM}(a) &= \left(\frac{a\Omega}{c} \right)^{-1} \sum_{\nu=3/2}^{\nu_0^{TM}} \nu \left[\Delta_0 \zeta_\nu^{TM}(s, 0^+) - \Delta_0 \zeta_\nu^{TM}(s, x) \right] \Big|_{s \rightarrow -1} = \\ &= \left(\frac{a\Omega}{c} \right)^{-1} \left\{ -\frac{1}{4n_<} \sum_{\nu \leq \nu_<} \nu^2 - \frac{1}{4n_>} \sum_{\nu \leq \nu_>} \nu^2 + \frac{a}{4n_2 R} \sum_{\nu \leq \nu_0^{TE}} \nu^2 + \frac{n_<^2}{\pi n_>} \left[\Pi \left(\frac{\pi}{2}, 1 - \frac{n_<^4}{n_>^4}, \sqrt{1 - \frac{n_<^2}{n_>^2}} \right) \sum_{\nu \leq \nu_<} \nu^2 + \right. \right. \\ &\quad \left. \left. + \sum_{\nu_< < \nu \leq \nu_>} \nu^2 \Pi \left(\arcsin \sqrt{\frac{\nu_>^2 - \nu^2}{\nu_>^2 - \nu_<^2}}, 1 - \frac{n_<^4}{n_>^4}, \sqrt{1 - \frac{n_<^2}{n_>^2}} \right) \right] \right\}. \end{aligned} \quad (57)$$

For simplicity, let us assume that the refraction indices are such that $(\nu_< - 1/2)$ and $(\nu_> - 1/2)$ are both integers. This does not lead to any loss of generality in the result we are looking for since, as in the case of the bulk contributions previously worked out, the fractional parts $\alpha_k = n_k x - [n_k x]$ have no effects on the leading terms of the sums for $(a\Omega/c \gg 1)$.

Almost all the sums appearing in the right hand side of Eqs. (56) and (57) can be trivially solved, since

$$\sum_{\nu=3/2}^{\nu_f} \nu^2 = \frac{\nu_f^3}{3} + \mathcal{O}(\nu_f^2). \quad (58)$$

The exception are the last terms appearing in those equations. Once again, these contributions can be approximated by means of the Euler - Maclaurin summation formula.

In so doing, one should remark that the functions in the argument of these sums, say $f(\nu)$, vanish (in both cases) as a square root at $\nu_<$ and at $\nu_>$. It is sufficient for our purposes to subtract a function $g(\nu)$ behaving the same way, in order to obtain a difference with a bounded positive first derivative, to which we can apply the Euler - Maclaurin formula [35],

$$\begin{aligned} \sum_{\nu=\nu_<+1}^{\nu_>} (f(\nu) - g(\nu)) &= \int_{\nu_<+1}^{\nu_>} (f(\nu) - g(\nu)) d\nu + \\ &+ (f(\nu_< + 1) - g(\nu_< + 1)) + \int_{\nu_<+1}^{\nu_>} (x - [x]) (f'(x) - g'(x)) dx. \end{aligned} \quad (59)$$

One can easily verify that the remainder (the last term in the right hand side) is $\mathcal{O}(a\Omega/c)^2$. On the other hand, the sum $\sum_{\nu=\nu_<+1}^{\nu_>} g(\nu)$ can be solved in terms of Hurwitz ζ -functions.

Gathering these results together we get

$$\frac{1}{\hbar \Omega} \Delta_0 E^{TE}(a) = \mathcal{O}\left(\frac{a \Omega}{c}\right) - \frac{1}{12} \left(\frac{n_2 R \Omega}{c}\right)^2 \left\{1 - 3 k_1^{TE} \left(\frac{n_2 R \Omega}{c}\right)^{-2/3}\right\} + \mathcal{O}\left(\frac{R \Omega}{c}\right) \quad (60)$$

for the TE modes, and

$$\begin{aligned} \frac{1}{\hbar \Omega} \Delta_0 E^{TM}(a) = & -\frac{1}{12} \frac{(n_{>}^2 - n_{<}^2)^2}{n_{>}^2 + n_{<}^2} \left(\frac{a \Omega}{c}\right)^2 + \mathcal{O}\left(\frac{a \Omega}{c}\right) + \\ & + \frac{1}{12} \left(\frac{n_2 R \Omega}{c}\right)^2 \left\{1 - 3 k_1^{TM} \left(\frac{n_2 R \Omega}{c}\right)^{-2/3}\right\} + \mathcal{O}\left(\frac{R \Omega}{c}\right) \end{aligned} \quad (61)$$

for the TM case.

Notice that for the TE modes there are no surface contributions coming from the interphase between dielectrics, in agreement with the results in [22] for nonmagnetic media. On the other hand, there is a negative surface contribution from the TM modes, vanishing for $n_1 = n_2$. Moreover, in both the TE and TM cases, there are surface contributions corresponding to the external boundary ($\sim R^2$), which differ in sign and cancel out when added. This is in accordance with the fact that Dirichlet and Neumann boundary conditions contribute with opposite surface terms [22] (see Section II). Finally, there are also corrections which depend on fractional powers of R , which are induced by the cut-off imposed (see Eqs. (43) and (45)).

Notice that the procedure followed to evaluate the Casimir energy can be continued up to any given order in the asymptotic expansion in Eq. (23), to get the result up to the corresponding order in powers of $(a\Omega/c)^{-1}$.

V. THE CASIMIR ENERGY

A. The electromagnetic vacuum pressure on the bubble

Let us stress again that the contributions depending on R in the right hand side of Eqs. (55) (and the corresponding result for the TM modes), (60) and (61) are independent of a and n_1 . They are exactly canceled if one refers the energy to that of the medium with n_2 filling completely the interior of the external sphere, by subtracting the same expressions with $a = 0$. In this way, any reference to the exterior radius R disappears and one obtains

$$\frac{1}{\hbar \Omega} \mathcal{E}(a) = \frac{(n_1^3 - n_2^3)}{6 \pi} \left(\frac{a \Omega}{c}\right)^3 - \frac{1}{12} \frac{(n_1^2 - n_2^2)^2}{n_1^2 + n_2^2} \left(\frac{a \Omega}{c}\right)^2 + \mathcal{O}\left(\frac{a \Omega}{c}\right). \quad (62)$$

The second term in the right hand side of Eq. (62) (a surface contribution, qualitatively similar to the one obtained in [32]) is negative, while the behavior of the first one (a volume term) depends on the sign of $(n_1^3 - n_2^3)$. For a large cut-off, the volume contribution is dominant, in agreement with the claim in [4,23,27,28].

As expected, $\mathcal{E}(a) = 0$ for $n_1 = n_2$. If, for example, $n_2 > n_1$, then $\mathcal{E}(a)$ is a negative function, monotonically decreasing with a . But, as remarked in [29], the values of $\mathcal{E}(a)$ for different bubble radius refer to configurations with different amounts of material media, and are not directly comparable.

Instead, we will retain the whole dependence of the vacuum energy with R , as obtained in Eqs. (55), (60) and (61), in order to allow for a variation of the refraction indices with the volume of the bubble, while keeping the number of molecules of each dielectric constant. This condition is equivalent to demanding that [29]

$$[n_1(a)^2 - 1] a^3 = \text{constant}, \quad [n_2(a)^2 - 1] (R^3 - a^3) = \text{constant}, \quad (63)$$

which implies that

$$n_1'(a) = -\frac{3}{2a} \frac{(n_1(a)^2 - 1)}{n_1(a)}, \quad n_2'(a) = \frac{3a^2}{2(R^3 - a^3)} \frac{(n_2(a)^2 - 1)}{n_2(a)}. \quad (64)$$

These derivatives, replaced in the expression of the pressure acting on the boundary between dielectrics due to the electromagnetic field,

$$\begin{aligned}
P(a) &\equiv \frac{-1}{4\pi a^2} \left[\frac{\partial}{\partial a} + n'_1(a) \frac{\partial}{\partial n_1} + n'_2(a) \frac{\partial}{\partial n_2} \right] E_{Cas.}(a) \\
&= \frac{-1}{4\pi a^2} \frac{d}{da} (\Delta_1 E^{TE}(a) + \Delta_1 E^{TM}(a) + \Delta_0 E^{TE}(a) + \Delta_0 E^{TM}(a) + \dots),
\end{aligned} \tag{65}$$

straightforwardly lead to

$$\begin{aligned}
\frac{P(a)}{\hbar \Omega} \left(\frac{\Omega}{c} \right)^{-3} &= \frac{-1}{16\pi^2} \left\{ \left[n_2(a)^3 - n_1(a)^3 \right] - 3 \left[n_2(a) - n_1(a) \right] \right\} + \frac{1}{24\pi} \left(\frac{a\Omega}{c} \right)^{-1} \left\{ \frac{\left(n_1(a)^2 - n_2(a)^2 \right)^2}{n_1(a)^2 + n_2(a)^2} + \right. \\
&\quad \left. + \frac{3}{2} \frac{\left(n_1(a)^2 - 1 \right) \left(n_2(a)^2 - n_1(a)^2 \right) \left(n_1(a)^2 + 3n_2(a)^2 \right)}{\left(n_1(a)^2 + n_2(a)^2 \right)^2} \right\} + \mathcal{O} \left(\frac{a\Omega}{c} \right)^{-2} + \mathcal{O} \left(\frac{R\Omega}{c} \right)^{-2/3},
\end{aligned} \tag{66}$$

where the limit $R \rightarrow \infty$ can be safely taken.

Notice that, even though $n_2(a)$ has a tiny derivative ($\sim R^{-3}$), it enters in a term with a large coefficient ($\sim R^3$), thus giving a finite contribution to $P(a)$. On the other hand, those terms containing lower powers of R do not contribute in the limit $R \rightarrow \infty$. Therefore, the expression obtained for $P(a)$ is presumably independent of the boundary conditions imposed on the field at $r = R$.

The first term in the right hand side of Eq. (66), coming from the bulk contribution to the vacuum energy (Eq. (55)), is clearly dominant for $a\Omega/c \gg 1$, while the second term, coming from the surface contributions (Eq. (61)) is less significant in this region. The pressure $P(a)$ behaves in the following way, depending on the values of the refraction indices: As expected, it vanishes for $n_2 = n_1$, and its derivative with respect to the exterior index is negative,

$$\frac{\partial P(a)}{\partial n_2} = -\frac{3}{16\pi^2} (n_2^2 - 1) + \mathcal{O} \left(\frac{a\Omega}{c} \right)^{-1} < 0, \tag{67}$$

for $n_2 > 1$. Therefore, for $n_2 > n_1$, $P(a) < 0$ and the bubble tends to shrink, while for $n_2 < n_1$, $P(a) > 0$ and the bubble tends to expand. In this way, we arrive at the nice picture of a dielectric tending to fill empty space, even for a vanishing electric field.

B. The Casimir Energy

The Casimir energy as a function of the bubble's radius (for given amounts of dielectric materials) can be obtained by integrating $-P(a)$ with respect to the bubble's volume. As remarked above, $n_2(a)$ is essentially constant, since this dielectric has a very large available volume (see Eq. (64)). On the other hand, when the bubble originally filled up with a dielectric of index n_1 is expanded from a volume $V_0 = 4\pi a_0^3/3$ to a volume $V = 4\pi a^3/3$, one finally gets a refraction index given by (see Eq. (63))

$$n_1(a) = \sqrt{1 + (n_1^2 - 1) \frac{V_0}{V}}. \tag{68}$$

Retaining only the dominant term in the expression of the pressure we get

$$\begin{aligned}
E_{Cas.}(a) - E_{Cas.}(a_0) &= - \int_{V_0}^V P(a) dV = \\
&= \hbar \Omega \left(\frac{\Omega}{c} \right)^3 \left\{ \frac{1}{16\pi^2} (n_2^3 - 3n_2) (V - V_0) + \frac{1}{8\pi^2} \left[V \left(1 + (n_1^2 - 1) \frac{V_0}{V} \right)^{\frac{3}{2}} - n_1^3 V_0 \right] \right\} \left(1 + \mathcal{O} \left(\frac{a\Omega}{c} \right)^{-1} \right).
\end{aligned} \tag{69}$$

Notice that, up to this first order, the dependence on n_1 and n_2 appears in separate terms.

This result is more easily analyzed in the case of a bubble containing a dilute medium, i.e., when $n_1^2 - 1 = \epsilon_1 - 1 \ll 1$. In this case, we get

$$E_{Cas.}(a) - E_{Cas.}(a_0) = \hbar\Omega \left(\frac{\Omega}{c}\right)^3 \left\{ \frac{(n_2 - 1)^2 (2 + n_2)}{16\pi^2} (V - V_0) - \frac{3(n_1^2 - 1)^2}{64\pi^2} \frac{V_0}{V} (V - V_0) + \mathcal{O}(n_1^2 - 1)^3 \right\}. \quad (70)$$

If the first term is dominant (with $n_2 > 1$) the Casimir energy increases with the volume. But if, for example, the bubble's exterior contains just vacuum ($n_2 = 1$), the first term in the right hand side of Eq. (70) vanishes, the second (with a negative coefficient quadratic in $(\epsilon_1 - 1)$) is the dominant one, and the Casimir energy decreases with the volume as V_0/V .

C. Application to Sonoluminescence

One reason for the great attention devoted to the study of Casimir energies of dielectric media with spherical symmetry is the suggestion made by J. Schwinger [4] that the Casimir effect might have a key role in the explanation of the peculiar phenomenon of sonoluminescence (consisting in the transduction of sound into light, see [5] and references therein).

This phenomenon (for a review, see [5]) is characterized by the fact that the energy enters a fluid as a sound wave (of $\sim 26\text{ kHz}$) which induces on a single gas bubble (air with some proportion of a noble gas), trapped in a velocity node, the emission of flashes of light in synchrony with the sound.

The flash is emitted at the end of the sudden collapse the bubble suffers each acoustic cycle. This collapse takes around $4\mu\text{s}$, reduces the radius of the bubble from around $45\mu\text{m}$ by a factor 10^{-1} , and makes its surface to reach supersonic velocities.

The violent deceleration of the bubble at the minimum radius is also accompanied by the emission of an outgoing acoustic pulse. After that, the bubble stays dead waiting for the next cycle. The posterior expansion occurs on hydrodynamic time scales, during the rarefaction half-cycle of the pressure (with some inertia making the bubble to reach its maximum size when the external sound field has already turned compressive).

Each flash of light contains about one million of visible photons, and is approximately spherically symmetric. Its duration (less than 50 ps) is a hundred times shorter than the shortest (visible) lifetime of an excited state of the hydrogen atom. The energy of the photons ranges up to 6.5 eV (higher frequency photons cannot propagate through water), and the power of the flash can reach 100 mW .

If the light were emitted from a region of atomic dimensions, a comparison of the flash energy with the average acoustic energy delivered to an atom of the fluid by the sound wave leads to conclude that a concentration of energy by twelve orders of magnitude should have occurred.

This phenomenon is visible to the naked eye in a darkened room as a starlike light.

Even though the hydrodynamical description of the collapse of the bubble and its posterior expansion is well understood [5], the mechanism through which part (about 0.01%) of the energy supplied by the sound is emitted as a flash of light is unknown and appears to be very complex [5].

Nevertheless, Schwinger suggested that the Casimir effect might be the underlying physics behind sonoluminescence, in the sense that the difference in the (static) electromagnetic zero point energy due to the change of the bubble's radius would be the available energy to be emitted as photons at the end of the bubble's collapse. In spite of the simplicity of this proposal, there has been no agreement about how to evaluate this change in the Casimir energy, and different approaches have led to controversial conclusions [21–24, 17, 25–28].

In particular, the presence of singularities renders the physical interpretation of the energy finite parts difficult. However, these inconveniences may have their origin in the fact that the models usually employed in describing dielectric media mostly do not incorporate a realistic frequency dependent dispersion relation, then leading to an inadequate ultraviolet behavior.

In a recent paper [31], a nonmagnetic dielectric ball with a frequency dependent permittivity (a high energy approximation to the Drude model), has been considered. It has been shown that a very simple pole structure results for the corresponding ζ -function, and only a volume energy counterterm (to be absorbed in the mass density of the material) is needed to render the Casimir energy finite. Neither surface nor curvature counterterms are necessary.

With the ultraviolet behavior under control, it makes sense to analyze the finite parts of the Casimir energy for realistic media. In this context, the analysis of the simple model under consideration is a step in the direction of incorporating finite frequency contributions.

For a spherical bubble of gas surrounded by water, we can take $n_1 = 1$ and $n_2 = 4/3$. In this case, the pressure (Eq. (66)) reduces to

$$\frac{P(a)}{\hbar \Omega} \left(\frac{\Omega}{c} \right)^{-3} = \frac{-5}{216 \pi^2} + \frac{49}{5400 \pi} \left(\frac{a \Omega}{c} \right)^{-1} + \mathcal{O} \left(\frac{a \Omega}{c} \right)^{-2} \simeq -2 \times 10^{-3}, \quad (71)$$

approximately a negative constant if $(a \Omega/c) \gg 1$, while the difference of Casimir energies (Eq. (69)) is

$$\frac{E_{Cas.}(a_0) - E_{Cas.}(a)}{\hbar \Omega} \left(\frac{\Omega}{c} \right)^{-3} \simeq \frac{5}{216 \pi^2} (V_0 - V) \simeq 2 \times 10^{-3} (V_0 - V). \quad (72)$$

Let us now consider an initial radius $a_0 \simeq 45 \mu m$, and a final one $a = a_0/10$. Then $V_0 - V = V_0(1 - 10^{-3}) \simeq V_0$.

Firstly, we will estimate the difference of Casimir energies by equating it to the emitted energy. Assuming that the flash has one million photons with an average energy of $5 eV$, we get $(E_{Cas.}(a_0) - E_{Cas.}(a)) \simeq 5 \cdot 10^6 eV$. Equation (72) then gives $(a_0 \Omega/c) \simeq 608$, which justifies the asymptotic expansion we have employed. The frequency cut-off turns out to be $\Omega \simeq 4 \cdot 10^{15} 1/s$, equivalent to a (visible) energy of around $2.6 eV$. Notice that the refraction index of water becomes essentially 1 at frequencies of the order $10^{16} 1/s$ (see Ref. [36], page 291). This cut-off corresponds to an electromagnetic pressure $P \simeq -2 \cdot 10^{-5} atm$, of a much smaller magnitude than the acoustic pressure on the bubble ($\sim 1 atm$) [5].

On the other hand, if we take instead as cut-off the frequency above which there is no propagation of photons in the water, $\Omega \simeq 10^{16} 1/s$ (corresponding to an energy of $6.5 eV$, with $a_0 \Omega/c \simeq 1490$), we get for the difference of Casimir energies $(E_{Cas.}(a_0) - E_{Cas.}(a)) \simeq 1.8 \cdot 10^8 eV$. The corresponding electromagnetic pressure is $P \simeq -7.5 \cdot 10^{-4} atm$.

Although obtained in the framework of a simplified model which ignores the complicated refraction index's dependence on the frequency, these results support Schwinger's proposal about the role the Casimir energy plays in sonoluminescence: It can behave as a reservoir of energy for the flash emission, which is feeded during the expansion of the bubble.

VI. SUMMARY AND CONCLUSIONS

In this paper we have considered a simple model of dielectric media, for which the permittivity and permeability are taken as constants up to a common cut-off frequency Ω , above which they take the values corresponding to the vacuum. This assumption reflects itself in frequency dependent boundary conditions for the electromagnetic field at the interphase between dielectric materials, which becomes transparent for frequencies greater than the cut-off.

For simplicity, we have limited our attention to nonmagnetic media, and studied the Casimir energy of a spherical dielectric of radius a and refraction index n_1 immersed in a second material of index n_2 , the whole contained in a large conducting concentric sphere of radius R .

In this context, the (divergent) contribution of the frequencies higher than Ω can be subtracted out by simply shifting the reference energy level. Indeed, it is independent of the (low frequency) refraction indices of both media, and also of the radius a of the internal sphere.

On the other hand, the contribution of the eigenfrequencies lower than Ω reduces just to two finite sums, for transversal electric (TE) and transversal magnetic (TM) modes respectively. For each angular momentum $l = \nu - 1/2$, these sums have been represented as differences of $\zeta_\nu^{TE}(x, s)$ and $\zeta_\nu^{TM}(x, s)$, the incomplete ζ -functions of the model (introduced in [33] for the case of a scalar field with a frequency dependent boundary condition).

In Appendix A we have derived the expression of the function $\Delta_\nu^{TE}(z)$, whose roots determines the TE eigenfrequencies for this configuration of material media. We have also proved that those among its zeroes lying in the open right half z -plane are all real and simple. The same is true for $\Delta_\nu^{TM}(z)$.

This fact allowed us to represent the incomplete ζ -functions as integrals on the complex plane, employing the Cauchy theorem. Finally, the uniform asymptotic Debye expansion for Bessel functions allowed for a systematic development of these ζ -functions, which facilitates the necessary analytic extensions.

In Section II we have retained as many terms in this approximation as necessary to isolate the singular pieces of the incomplete ζ -functions at $s = -1$. This has proved sufficient to evaluate the bulk and the first finite size contributions to the Casimir energy.

Since the lowest positive zero of $\Delta_\nu^{TE}(z)$ ($\Delta_\nu^{TM}(z)$) is a growing function of ν , in Section III we have determined ν_0^{TE} (ν_0^{TM}), the maximum value of ν for which there are eigenfrequencies smaller than or equal to Ω . Starting from the analytic extension of the incomplete ζ -functions to $s \simeq 0$, we have been able to show that ν_0^{TE} and ν_0^{TM} are linear functions of $(n_2 R \Omega/c)$ (with corrections depending on lower non-integer powers of this parameter, induced by the presence of a cut-off), which are independent of n_1 and a .

In Section IV we have shown that, for $(a\Omega/c) \gg 1$, the dominant contributions are volume terms (see Eq. (55)). There is a piece proportional to the bubble's volume, whose sign is determined by the difference $(n_1^3 - n_2^3)$, plus a positive term proportional to the volume of the accessible space, which depends neither on n_1 nor on a .

The second order in the Debye expansion produces the first finite size corrections to the Casimir energy, which are surface contributions (see Eqs. (60) and (61)). There is a negative term proportional to the bubble's surface, coming exclusively from the TM modes. There are also surface terms corresponding to the external boundary, due to the TE and TM modes, which differ in sign and cancel out when added.

Finally, there are also corrections proportional to non-integer powers of $(n_2 R \Omega/c)$ (the power $\frac{7}{3}$ and lower) induced by the relation between ν_0^{TE} (ν_0^{TM}) and R . These corrections, however, have no consequences in the $R \rightarrow \infty$ limit.

These results are analyzed in Section V.

Firstly, we have considered the difference between the Casimir energy so evaluated and the one corresponding to the second medium filling completely the interior of the external sphere. As given in Eq. (62), this difference is independent of R . Its behavior with respect to a depends on the values of n_1 and n_2 but, as remarked above, its values for different radius refer to different amounts of material media.

In order to determine the force acting on the interphase between dielectrics, one should rather impose the conservation of the number of particles in each medium. This condition leads to a variation of the refraction indices with a . In this way, we arrived at an expression for the electromagnetic pressure $P(a)$ as a function of a (see Eq. (66)), where we could safely take the $R \rightarrow \infty$ limit to obtain the first terms of an expansion in $(a\Omega/c)^{-1} \ll 1$. $P(a)$ so constructed vanishes for $n_2 = n_1$, and has a negative derivative with respect to n_2 . Therefore, it is negative for $n_2 > n_1$, and tends to compress the bubble.

This pressure can be integrated to get the variation of the Casimir energy with respect to the bubble's volume, for given amounts of material media (see Eq. (69)).

When considering models of dielectrics with constant refraction indices, the presence of divergencies makes it difficult to give a physical interpretation to the finite part of the vacuum energy, which cannot be isolated from the singular one. But when a realistic ultraviolet behavior is assumed, the singularities can be removed with a single volume energy counterterm [31], to be absorbed in the mass density of the material. Neither surface nor curvature counterterms are needed to render the Casimir energy finite. With the ultraviolet behavior under control, one can worry about the finite frequency contributions in realistic models.

The present paper, where low frequency refraction indices are modeled as constants up to the cut-off Ω , can be considered as a step in this direction.

Finally, we have applied the expressions found to a situation of interest for the phenomenon of sonoluminescence. Our results support Schwinger's proposal about the role the Casimir energy plays in the transduction of sound into light.

Indeed, for the case of a spherical bubble of gas surrounded by water we can assume $n_1 = 1$ and $n_2 = 4/3$. For a typical sonoluminescing bubble, the ambient radius is $a \simeq 4.5 \mu m$ (one tenth of its maximum radius).

If we, moreover, estimate the difference in vacuum energies as the energy of a flash of light, we get from Eq. (71) an approximately negative constant electromagnetic pressure (favoring the collapse of the bubble, although of a magnitude much less than the acoustic pressure). In these conditions, the cut-off Ω turns out to be in the region of the visible spectrum, large enough to justify the approach followed in this paper, and not very far from the region where the refraction index of water becomes essentially 1.

On the other side, if the frequency cut-off is imposed by hand where the propagation of light in water is no longer possible, then the change in the vacuum energy due to the collapse is about forty times the energy typically emitted in each flash.

These results, obtained in the framework of a *realistic* dielectric model (which otherwise ignores the complicated refraction index's dependence on the frequency), clearly seems to support Schwinger's ideas about the role the Casimir energy plays in sonoluminescence: It grows with the bubble's volume by an amount comparable with the flash energy, which is therefore available to be emitted as light at the end of the collapse.

Of course, this does not explain why the flash is emitted in such a short time at the end of the sudden collapse of the bubble. One could speculate about the formation of an excited electromagnetic field state, which would be induced to decay through some mechanism related to the strong deceleration stopping the bubble at its minimum radius.

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In this Appendix we will study the solutions of Eq. (8), subject to the boundary conditions for the TE modes, Eqs. (6). We will derive the expression of the function $\Delta_{l+1/2}^{TE}(z)$ in Eq. (9) (whose roots determine the eigenfrequencies), and show that all its zeroes lying in the open right half z -plane are real and simple, a condition allowing for the integral representation in Eq. (15). In particular, this implies that the only degeneracy of the eigenfrequencies is $(2l+1)$, due to the spherical symmetry of the problem.

To this end, it is convenient to define

$$s = s(r) = \begin{cases} \mu_1 r, & r \leq a \\ \mu_2 r + a_1 - a_2, & r > a \end{cases} \quad (\text{A1})$$

with $a_{1,2} = \mu_{1,2} a$. Then, expressing $\varphi(s) \equiv r f_l(r)$ in terms of the new variable, taking into account that $\frac{d\varphi}{dr} = \frac{ds}{dr} \frac{d\varphi}{ds} = \mu_k \frac{d\varphi}{ds}$, with $k = 1$ ($k = 2$) for $r < a$ ($r > a$), and calling

$$\epsilon(s) = \epsilon_1 \Theta(a_1 - s) + \epsilon_2 \Theta(s - a_1), \quad \mu(s) = \mu_1 \Theta(a_1 - s) + \mu_2 \Theta(s - a_1), \quad (\text{A2})$$

we get, from Eq. (8), the differential equation (with discontinuous coefficients)

$$\hat{\mathcal{L}}_l \varphi(s) \equiv \frac{\mu(s)}{\epsilon(s)} \left\{ \frac{d^2}{ds^2} - \frac{l(l+1)}{[s - (a_1 - a_2)\Theta(s - a_1)]^2} \right\} \varphi(s) = -\frac{\omega^2}{c^2} \varphi(s), \quad (\text{A3})$$

for $s \neq a_1$. Here, $l = 1, 2, \dots$

Moreover, Eq. (6) results in the continuity conditions

$$\varphi(s = a_1^+) = \varphi(s = a_1^-), \quad \text{and} \quad \varphi'(s = a_1^+) = \varphi'(s = a_1^-). \quad (\text{A4})$$

So, we are looking for solutions of Eq. (A3) with a continuous first derivative, $\varphi(s) \in \mathcal{C}^1(\mathbf{R}^+)$.

In Subsection A 1 we will show that Eq. (A3) for this kind of functions, complemented with adequate boundary conditions, defines a self-adjoint operator. This excludes the possibility of non-real eigenvalues $-\omega^2/c^2$. The function $\Delta_\nu^{TE}(z)$ is obtained in Subsection A 2. Finally, in Subsection A 3 we will show that the nonvanishing zeroes of $\Delta_\nu^{TE}(z)$ are simple.

1. Self-adjointness

Let us consider the operator \mathcal{L}_l defined as the differential operator $\hat{\mathcal{L}}_l$ in the left hand side of Eq. (A3), with a domain restricted to $\mathcal{D}(\mathcal{L}_l) = \mathcal{C}_0^\infty[0, s_0] \subset \mathbf{L}^2(\mathbf{R}^+, (\epsilon/\mu)(s) ds)$, where $s_0 = \mu_2 R + a_1 - a_2$ (with $R > a$). Here $\mathcal{C}_0^\infty[0, s_0]$ is the space of functions with continuous derivatives of all orders and identically vanishing on some neighborhood of 0 and s_0 . Clearly, \mathcal{L}_l is symmetric on $\mathcal{D}(\mathcal{L}_l)$,

$$(\varphi_1, \mathcal{L}_l \varphi_2)_{\mathbf{L}^2(\mathbf{R}^+, (\epsilon/\mu)(s) ds)} = (\mathcal{L}_l \varphi_2, \varphi_1)_{\mathbf{L}^2(\mathbf{R}^+, (\epsilon/\mu)(s) ds)}. \quad (\text{A5})$$

It is straightforward to show that its adjoint [37], \mathcal{L}_l^\dagger , is defined on the subspace $\mathcal{D}(\mathcal{L}_l^\dagger) \subset \mathbf{L}^2(\mathbf{R}^+, (\epsilon/\mu)(s) ds)$ containing those functions $\psi(s)$ with an absolutely continuous first derivative, and such that $\psi''(s) - V_l(s)\psi(s) \in \mathbf{L}^2([0, \delta], ds)$, for $\delta > 0$ (without requiring any further boundary condition). Moreover, for $\psi \in \mathcal{D}(\mathcal{L}_l^\dagger)$, the action of \mathcal{L}_l^\dagger reduces to the application of the differential operator $\hat{\mathcal{L}}_l$.

In order to determine the deficiency indices of \mathcal{L}_l (defined as $n_\pm(\mathcal{L}_l) = \dim \text{Ker}(\mathcal{L}_l^\dagger \mp i)$, see [37]) one must look for the linearly independent solutions of $\mathcal{L}_l^\dagger \psi(s) = \pm i \psi(s)$ in $\mathcal{D}(\mathcal{L}_l^\dagger)$.

Notice that the second derivatives of such functions are continuous for $s \neq a_1$. Moreover, if $\psi(s)$ is a solution of $\mathcal{L}_l^\dagger \psi(s) = +i \psi(s)$, then its complex conjugate $\psi(s)^*$ is a solution of $\mathcal{L}_l^\dagger \psi(s)^* = -i \psi(s)^*$. This implies that \mathcal{L}_l has equal deficiency indices, $n_-(\mathcal{L}_l) = n_+(\mathcal{L}_l)$, thus admitting self-adjoint extensions [37].

In fact, it can be seen that the equation

$$\mathcal{L}_l^\dagger \psi(s) = \imath \psi(s) \quad (\text{A6})$$

has a unique (up to a constant factor) solution in $\mathcal{D}(\mathcal{L}_l^\dagger)$. Moreover, it vanishes at the origin. Therefore (see [37]), there exists a one parameter family of self-adjoint extensions of \mathcal{L}_l , which are in a one to one correspondence with the unitary maps from $\text{Ker}(\mathcal{L}_l^\dagger - \imath)$ onto $\text{Ker}(\mathcal{L}_l^\dagger + \imath)$, given by $\psi(s) \rightarrow \alpha \psi(s)^*$, where $\psi(s)$ is the solution of Eq. (A6), and $\alpha \in \mathbf{C}$ with $|\alpha| = 1$.

Each essentially self-adjoint extension $\mathcal{L}_l^{(\alpha)}$ is defined on a domain given by [37]

$$\mathcal{D}(\mathcal{L}_l^{(\alpha)}) = \{\phi(s) = \varphi(s) + \beta [\psi(s) + \alpha \psi(s)^*]; \varphi(s) \in \mathcal{C}_0^\infty[0, s_0], \beta \in \mathbf{C}\}, \quad (\text{A7})$$

with $\mathcal{L}_l^{(\alpha)}$ acting on $\phi(s) \in \mathcal{D}(\mathcal{L}_l^{(\alpha)}) \subset \mathcal{D}(\mathcal{L}_l^{(\dagger)})$ as

$$\mathcal{L}_l^{(\alpha)} \phi(s) = \mathcal{L}_l^\dagger \phi(s) = \mathcal{L}_l \varphi(s) + \imath \beta (\psi(s) - \alpha \psi(s)^*). \quad (\text{A8})$$

In particular, notice that $\phi(0) = 0$.

Each essentially self-adjoint extension of \mathcal{L}_l can also be characterized by the homogeneous boundary condition the functions in $\mathcal{D}(\mathcal{L}_l^{(\alpha)})$ satisfy at $s = s_0$. In fact, for all $\beta \neq 0$, Eq. (A7) implies that

$$\phi'(s_0) + c(\alpha)\phi(s_0) = 0, \quad (\text{A9})$$

with $c(\alpha) \in \mathbf{R} \cup \{\infty\}$ (condition also satisfied for $\beta = 0$).

2. The eigenfrequencies

As seen before, we should impose on the fields a local homogeneous boundary condition at $s = s_0$ (i.e. $r = R > a$). This determines the functions to be included in the domain of the relevant operator, Eq. (A7).

We choose to enclose the system within a large conducting sphere of radius R , obtaining the Dirichlet condition at $s = s_0$ for the functions in the domain of an essentially self-adjoint extension of \mathcal{L}_l which we call $\mathcal{L}_l^{(D)}$:

$$E_{\theta, \phi} \Big|_{r=R} = 0 \quad \Rightarrow \quad \phi(s) \Big|_{s=s_0} = 0, \quad \forall l \geq 1. \quad (\text{A10})$$

So, the eigenfunctions ϕ_ω of $\mathcal{L}_l^{(D)}$ satisfy the differential equation (A3), $\hat{\mathcal{L}}_l \phi_\omega(s) = -(\omega/c)^2 \phi_\omega(s)$ for $s \neq a_1$, and the boundary and continuity conditions

$$\begin{aligned} \phi_\omega(s)|_{s=0} &= 0, & \phi_\omega(s)|_{s=s_0} &= 0, \\ \phi_\omega(s)|_{s=a_1^+} &= \phi_\omega(s)|_{s=a_1^-}, & \phi'_\omega(s)|_{s=a_1^+} &= \phi'_\omega(s)|_{s=a_1^-}. \end{aligned} \quad (\text{A11})$$

This reduces the problem to looking for functions with a continuous second derivative for $s \neq a_1$, which satisfy

$$\left\{ \frac{d^2}{dz_1^2} + \left[1 - \frac{l(l+1)}{z_1^2} \right] \right\} \phi_\omega(s) = 0, \quad (\text{A12})$$

with $z_1 = s(\omega/c)\sqrt{\epsilon_1/\mu_1}$, for $s < a_1$, which are solutions of the same equation with $z_1 \rightarrow z_2 = (s-a_1+a_2)(\omega/c)\sqrt{\epsilon_2/\mu_2}$, for $s > a_1$, and satisfy the boundary and continuity conditions stated in Eq. (A11). Therefore,

$$\begin{aligned} \phi_\omega(s) &= A_1 \mathcal{J}_{l+1/2}(z_1) + B_1 \mathcal{Y}_{l+1/2}(z_1), \quad \text{for } s < a_1, \\ \phi_\omega(s) &= A_2 \mathcal{J}_{l+1/2}(z_2) + B_2 \mathcal{Y}_{l+1/2}(z_2), \quad \text{for } a_1 < s < s_0, \end{aligned} \quad (\text{A13})$$

where $\mathcal{J}_{l+1/2}(z) = z j_l(z)$ and $\mathcal{Y}_{l+1/2}(z) = z y_l(z)$ are the Riccati - Bessel functions, $j_l(z)$ and $y_l(z)$ being the spherical Bessel functions.

Since

$$\mathcal{J}_{l+1/2}(z) = \frac{z^{l+1}}{\Gamma(2(l+1))} (1 + \mathcal{O}(z^2)), \quad \mathcal{Y}_{l+1/2}(z) = -\frac{\Gamma(2l)}{z^l} (1 + \mathcal{O}(z^2)), \quad (\text{A14})$$

the condition $\phi(0) = 0$ implies that $B_1 = 0, \forall l$.

To ensure that $\phi(s_0) = 0$ we can take

$$A_2 = \mathcal{Y}_{l+1/2}(\bar{z}_0), \quad B_2 = -\mathcal{J}_{l+1/2}(\bar{z}_0), \quad (\text{A15})$$

with $\bar{z}_0 = (s_0 - a_1 + a_2)(\omega/c)\sqrt{\epsilon_2/\mu_2}$.

Finally, the continuity conditions at $s = a_1$ give

$$\begin{aligned} A_1 \mathcal{J}_{l+1/2}(\bar{z}_1) &= \mathcal{Y}_{l+1/2}(\bar{z}_0) \mathcal{J}_{l+1/2}(\bar{z}_2) - \mathcal{J}_{l+1/2}(\bar{z}_0) \mathcal{Y}_{l+1/2}(\bar{z}_2), \\ A_1 \mathcal{J}'_{l+1/2}(\bar{z}_1) &= \sqrt{\frac{\epsilon_2 \mu_1}{\epsilon_1 \mu_2}} \left\{ \mathcal{Y}_{l+1/2}(\bar{z}_0) \mathcal{J}'_{l+1/2}(\bar{z}_2) - \mathcal{J}_{l+1/2}(\bar{z}_0) \mathcal{Y}'_{l+1/2}(\bar{z}_2) \right\}, \end{aligned} \quad (\text{A16})$$

where $\bar{z}_{1,2} = a(\omega/c)\sqrt{\epsilon_{1,2}\mu_{1,2}}$.

So, defining $z = a\omega/c$, the eigenfrequencies are determined by the zeroes of the function

$$\begin{aligned} \Delta_{l+1/2}^{TE}(z) &= \mathcal{J}_{l+1/2}(\bar{z}_1) \left\{ \mathcal{Y}_{l+1/2}(\bar{z}_0) \mathcal{J}'_{l+1/2}(\bar{z}_2) - \mathcal{J}_{l+1/2}(\bar{z}_0) \mathcal{Y}'_{l+1/2}(\bar{z}_2) \right\} - \\ &\quad - \xi \mathcal{J}'_{l+1/2}(\bar{z}_1) \left\{ \mathcal{Y}_{l+1/2}(\bar{z}_0) \mathcal{J}_{l+1/2}(\bar{z}_2) - \mathcal{J}_{l+1/2}(\bar{z}_0) \mathcal{Y}_{l+1/2}(\bar{z}_2) \right\}, \end{aligned} \quad (\text{A17})$$

where $\xi = \sqrt{\frac{\epsilon_1 \mu_2}{\epsilon_2 \mu_1}}$.

Notice that every zero of $\Delta_{l+1/2}^{TE}(z)$ determines an eigenvector of $\mathcal{L}_l^{(D)}$. Indeed, the function $\phi_\omega(s)$ constructed as above is in $\mathcal{D}(\mathcal{L}_l^{(D)})$ and satisfies Eq. (A3), $-(\omega/c)^2$ being the corresponding eigenvalue.

Consequently, all the zeroes of $\Delta_{l+1/2}^{TE}(z)$ are either real or purely imaginary, since the operator $\mathcal{L}_l^{(D)}$ is essentially self-adjoint for every $l = 1, 2, \dots$

3. The multiplicities

The condition determining the eigenvalues can also be understood from the following point of view. First notice that, for a given $k \in \mathbf{C}$, the differential equation

$$\hat{\mathcal{L}}_l \phi(s; k) = -k^2 \phi(s; k) \quad (\text{A18})$$

has, for $s \in (0, a_1)$ and for $s \in (a_1, s_0)$, two linearly independent solutions in \mathcal{C}^∞ . So, a given function which is a solution at one side of the point $s = a_1$ can be continued as a solution to the other side, to obtain a $\mathcal{C}^1(0, s_0)$ function with a piecewise continuous second derivative.

Now, let us call $\varphi(s; k)$ a square integrable non-trivial solution of Eq. (A18) satisfying $\varphi(0; k) = 0$ (unique up to a constant factor, see Eq. (A14)):

$$\varphi(s) = \mathcal{J}_{l+1/2}(z_1), \quad \text{for } 0 < s < a_1, \quad (\text{A19})$$

with $z_1 = s k \sqrt{\epsilon_1/\mu_1}$, and

$$\varphi(s) = A(k) \mathcal{J}_{l+1/2}(z_2) + B(k) \mathcal{Y}_{l+1/2}(z_2), \quad \text{for } a_1 < s < s_0, \quad (\text{A20})$$

with $z_2 = (s - a_1 + a_2) k \sqrt{\epsilon_2/\mu_2}$. The coefficients $A(k)$ and $B(k)$ are determined by the continuity conditions at $s = a_1$,

$$\begin{aligned} A(k) \mathcal{J}_{l+1/2}(\bar{z}_2) + B(k) \mathcal{Y}_{l+1/2}(\bar{z}_2) &= \mathcal{J}_{l+1/2}(\bar{z}_1), \\ A(k) \mathcal{J}'_{l+1/2}(\bar{z}_2) + B(k) \mathcal{Y}'_{l+1/2}(\bar{z}_2) &= \sqrt{\frac{\epsilon_1 \mu_2}{\epsilon_2 \mu_1}} \mathcal{J}'_{l+1/2}(\bar{z}_1), \end{aligned} \quad (\text{A21})$$

where $\bar{z}_{1,2} = a k \sqrt{\epsilon_{1,2}\mu_{1,2}}$.

Similarly, let $\chi(s; k)$ and $\rho(s; k)$ be $\mathcal{C}^1(0, s_0)$ functions satisfying Eq. (A18) for $s \neq a_1$, and the conditions

$$\begin{aligned}\chi(s_0; k) &= 0, \quad \chi'(s_0; k) \neq 0, \\ \rho(s_0; k) &\neq 0, \quad \rho'(s_0; k) = 0.\end{aligned}\tag{A22}$$

For $s > a_1$ they can be taken as

$$\begin{aligned}\chi(s; k) &= \mathcal{Y}_{l+1/2}(\bar{z}_0) \mathcal{J}_{l+1/2}(z_2) - \mathcal{J}_{l+1/2}(\bar{z}_0) \mathcal{Y}_{l+1/2}(z_2), \\ \rho(s; k) &= \mathcal{Y}'_{l+1/2}(\bar{z}_0) \mathcal{J}_{l+1/2}(z_2) - \mathcal{J}'_{l+1/2}(\bar{z}_0) \mathcal{Y}_{l+1/2}(z_2),\end{aligned}\tag{A23}$$

where $\bar{z}_0 = (s - a_1 + a_2) k \sqrt{\epsilon_2/\mu_2}$.

Since all these functions are $\mathcal{C}^1(0, s_0)$ solutions of Eq. (A18), the Wronskian of any two of them is a (k -dependent) constant (for $s < a_1$ and for $s > a_1$, and therefore for $0 < s < s_0$), which vanishes if and only if the selected functions are linearly dependent. In particular,

$$\begin{aligned}W[\rho(s; k), \chi(s; k)] &= \{\rho(s; k) \chi'(s; k) - \rho'(s; k) \chi(s; k)\}|_{s \rightarrow s_0} = \\ &= -k \sqrt{\frac{\epsilon_2}{\mu_2}} (W[\mathcal{J}_{l+1/2}(\bar{z}_0), \mathcal{Y}_{l+1/2}(\bar{z}_0)])^2 \neq 0, \quad \text{for } k \neq 0,\end{aligned}\tag{A24}$$

since $\mathcal{J}_{l+1/2}(z_2)$ and $\mathcal{Y}_{l+1/2}(z_2)$ are linearly independent solutions of Eq. (A18) for $s > a_1$.

Let us call

$$\begin{aligned}\eta(k) &\equiv W[\varphi, \chi] = \varphi(s; k) \chi'(s; k) - \varphi'(s; k) \chi(s; k), \\ \sigma(k) &\equiv W[\varphi, \rho] = \varphi(s; k) \rho'(s; k) - \varphi'(s; k) \rho(s; k).\end{aligned}\tag{A25}$$

Since there are only two linearly independent solutions of Eq. (A18), $\varphi(s; k)$ can be expressed as a linear combination of $\chi(s; k)$ and $\rho(s; k)$. In fact,

$$\begin{aligned}\eta(k) \rho(s; k) - \sigma(k) \chi(s; k) &= \{\varphi(s; k) \chi'(s; k) - \varphi'(s; k) \chi(s; k)\} \rho(s; k) - \\ &- \{\varphi(s; k) \rho'(s; k) - \varphi'(s; k) \rho(s; k)\} \chi(s; k) = W[\rho, \chi] \varphi(s; k).\end{aligned}\tag{A26}$$

Consequently, $\varphi(s; k)$ and $\chi(s; k)$ are proportional for a given k (and, therefore, are $\mathcal{C}^1(0, s_0)$ solutions of Eq. (A18) satisfying the Dirichlet boundary condition at $s = 0, s_0$) if and only if $\eta(k) = 0$.

Then, if $\eta(k_0) = 0$ we have

$$\varphi(s; k_0) \in \mathcal{D}(\mathcal{L}_l^{(D)}), \quad \text{and} \quad \mathcal{L}_l^{(D)} \varphi(s; k_0) = -k_0^2 \varphi(s; k_0).\tag{A27}$$

Since $\mathcal{L}_l^{(D)}$ is essentially self-adjoint, then $k_0^2 \in \mathbf{R}$, which implies that the zeroes of $\eta(k)$ are either real or purely imaginary.

Moreover, since $\eta(k)$ is independent of s , the Wronskian can be evaluated at $s = a_1$ obtaining

$$\eta(k) = W[\varphi, \chi](s = a_1) = \varphi(a_1^-; k) \chi'(a_1^+; k) - \varphi'(a_1^-; k) \chi(a_1^+; k),\tag{A28}$$

which is proportional to $\Delta_{l+1/2}^{TE}(a k)$, as can be easily verified (see Eqs. (A19), (A23) and (A17)).

We will finally show that the non-vanishing zeroes of $\eta(k)$ are simple. To this end, first notice that (as a function of s) $(\partial\varphi/\partial k)(s; k) \in \mathcal{C}^1(0, s_0)$, and $(\partial\varphi/\partial k)(s; k) \sim s^{l+1}$ for $s \rightarrow 0^+$. This can be easily shown from Eqs. (A19), (A20) and (A21). Moreover, from Eq. (A18) we get

$$\left\{ \frac{d^2}{ds^2} - V_l(s) + \frac{\epsilon(s)}{\mu(s)} k^2 \right\} \frac{\partial\varphi}{\partial k}(s; k) = -2k \frac{\epsilon(s)}{\mu(s)} \varphi(s; k)\tag{A29}$$

for $s \neq a_1$.

Similarly, $(\partial\chi/\partial k)(s; k) \in \mathcal{C}^1(0, s_0)$ and, from Eq. (A23), one can show that $(\partial\chi/\partial k)(s; k) \rightarrow 0$ for $s \rightarrow s_0$.

Let us now suppose that $k_0 \neq 0$ is a multiple zero of $\eta(k)$. Then, $\eta(k_0) = 0$ and $\eta'(k_0) = 0$. It follows from Eq. (A26) and (A24) that

$$\frac{\partial \varphi}{\partial k}(s; k_0) = K_1 \chi(s; k_0) + K_2 \frac{\partial \chi}{\partial k}(s; k_0), \quad (\text{A30})$$

for some constants K_1 and K_2 . Therefore, we also have that $(\partial \varphi / \partial k)(s; k_0) \rightarrow 0$ for $s \rightarrow s_0$.

Consequently, $(\partial \varphi / \partial k)(s; k_0) \in \mathcal{D}(\mathcal{L}_l^{(D)})$. Moreover, since $k_0^2 \in \mathbf{R}$, from Eq. (A29) we can write

$$\begin{aligned} -2k_0 \parallel \varphi(s; k_0) \parallel_{\mathbf{L}^2(\mathbf{R}^+, (\epsilon/\mu)(\mathbf{s}) \, d\mathbf{s})}^2 &= \left(\varphi(s; k_0), \left[\mathcal{L}_l^{(D)} + k_0^2 \right] \frac{\partial \varphi}{\partial k}(s; k_0) \right)_{\mathbf{L}^2(\mathbf{R}^+, (\epsilon/\mu)(\mathbf{s}) \, d\mathbf{s})} \\ &= \left(\left[\mathcal{L}_l^{(D)} + k_0^2 \right] \varphi(s; k_0), \frac{\partial \varphi}{\partial k}(s; k_0) \right)_{\mathbf{L}^2(\mathbf{R}^+, (\epsilon/\mu)(\mathbf{s}) \, d\mathbf{s})} = 0. \end{aligned} \quad (\text{A31})$$

But this is a contradiction, since $\varphi(s; k_0) \not\equiv 0$. Therefore, all the non-vanishing zeroes of $\eta(k)$ are simple.

An entirely similar analysis can be carried out for the TM case (now imposing the Neumann boundary condition at $r = R$), to conclude that the roots of $\Delta_{l+1/2}^{TM}(z)$ in Eq. (11) lying in the open right half z -plane are all real and simple.

APPENDIX B: DEBYE EXPANSIONS

For the TE modes, making use of the uniform asymptotic expansion [34] of the Bessel functions appearing in the expression of $\Delta_\nu^{TE}(z)$, Eq. (17), with $z \rightarrow i\nu t$ (for $\nu \gg 1$ and t fixed), and discarding terms vanishing exponentially for $R \rightarrow \infty$, we get Eqs. (23) and (24) where

$$D_{TE}^{(1)}(t) = \frac{1}{t} \left(\sqrt{1 + n_1^2 t^2} - \sqrt{1 + n_2^2 t^2} + \sqrt{1 + \frac{n_2^2 R^2 t^2}{a^2}} \right), \quad (\text{B1})$$

$$D_{TE}^{(0)}(t) = \frac{1}{2t} \left(\frac{1}{1 + n_1^2 t^2} + \frac{1}{1 + n_2^2 t^2} + \frac{a^2}{a^2 + n_2^2 R^2 t^2} - \frac{2}{\sqrt{1 + n_1^2 t^2} \sqrt{1 + n_2^2 t^2}} \right), \quad (\text{B2})$$

and

$$\begin{aligned} D_{TE}^{(-1)}(t) &= \frac{t a^2 n_2^2 R^2 (4a^2 - n_2^2 R^2 t^2) \sqrt{1 + \frac{n_2^2 R^2 t^2}{a^2}}}{8(a^2 + n_2^2 R^2 t^2)^3} \\ &\quad - \frac{t}{8(1 + n_1^2 t^2)^{\frac{5}{2}}(1 + n_2^2 t^2)^2} [-8n_2^2 + n_1^4 t^2 (n_2^4 t^4 - 3 - 10n_2^2 t^2) - 4n_1^2 (2 + 7n_2^2 t^2 + n_2^4 t^4)] \\ &\quad + \frac{t}{8(1 + n_1^2 t^2)^2(1 + n_2^2 t^2)^{\frac{5}{2}}} [n_1^4 n_2^2 t^4 (n_2^2 t^2 - 4) - n_2^2 (8 + 3n_2^2 t^2) - 2n_1^2 (4 + 14n_2^2 t^2 + 5n_2^4 t^4)]. \end{aligned} \quad (\text{B3})$$

Similarly, for the TM modes, the use of the uniform asymptotic Debye expansion of the Bessel functions leads to

$$\frac{d \log \Delta_\nu^{TM}(i\nu t)}{dt} = D_\nu^{TM}(t) + \mathcal{O}(\nu^{-2}), \quad (\text{B4})$$

where $D_\nu^{TM}(t)$ is given in Eq. (26) in terms of the algebraic functions

$$D_{TM}^{(1)}(t) = \frac{1}{t} \left(\sqrt{1 + n_1^2 t^2} - \sqrt{1 + n_2^2 t^2} + \sqrt{1 + \frac{n_2^2 R^2 t^2}{a^2}} \right), \quad (\text{B5})$$

$$\begin{aligned} D_{TM}^{(0)}(t) &= -\frac{n_1^2 t}{2(1 + n_1^2 t^2)} - \frac{n_2^2 t}{2(1 + n_2^2 t^2)} + \frac{n_1^2 n_2^2 t}{n_1^2 + n_2^2 + n_1^2 n_2^2 t^2} - \frac{1}{2t} \\ &\quad + \frac{n_1^2 n_2^2 t}{\sqrt{1 + n_1^2 t^2} \sqrt{1 + n_2^2 t^2} (n_1^2 + n_2^2 + n_1^2 n_2^2 t^2)} + \frac{n_2^2 R^2 t}{2(a^2 + n_2^2 R^2 t^2)}, \end{aligned} \quad (\text{B6})$$

and

$$\begin{aligned}
D_{TM}^{(-1)}(t) = & \frac{-(n_2^2 R^2 t (8 a^2 + n_2^2 R^2 t^2))}{8 a^4 \left(1 + \frac{n_2^2 R^2 t^2}{a^2}\right)^{\frac{5}{2}}} \\
& + \left(n_2^2 t \left(-4 n_2^4 + n_2^6 t^2 + n_1^8 t^4 (1 + n_2^2 t^2)^2 (8 + n_2^2 t^2) \right. \right. \\
& \left. \left. - 2 n_1^2 n_2^2 (6 + 11 n_2^2 t^2) + 2 n_1^6 t^2 (1 + n_2^2 t^2) (8 + 13 n_2^2 t^2 + 8 n_2^4 t^4) + \right. \right. \\
& \left. \left. n_1^4 n_2^2 t^2 (-3 + 6 n_2^2 t^2 + 14 n_2^4 t^4) \right) \right) / \left(8 (1 + n_1^2 t^2)^2 (1 + n_2^2 t^2)^{\frac{5}{2}} (n_1^2 + n_2^2 + n_1^2 n_2^2 t^2)^2 \right) - \\
& \left(n_1^2 t (8 n_2^6 t^2 (2 + n_2^2 t^2) + n_1^2 n_2^2 (-12 - 3 n_2^2 t^2 + 42 n_2^4 t^4 + 17 n_2^6 t^6) + \right. \\
& \left. n_1^6 (t^2 + 14 n_2^4 t^6 + 16 n_2^6 t^8 + n_2^8 t^{10}) + 2 n_1^4 (-2 + n_2^2 t^2 (-11 + 3 n_2^2 t^2 + 21 n_2^4 t^4 + 5 n_2^6 t^6))) \right) / \\
& \left(8 (1 + n_1^2 t^2)^{\frac{5}{2}} (1 + n_2^2 t^2)^2 (n_1^2 + n_2^2 + n_1^2 n_2^2 t^2)^2 \right).
\end{aligned} \tag{B7}$$

In eqs. (B5-B7) we have also discarded those contributions vanishing exponentially for $R \rightarrow \infty$, which come from those terms containing $K_\nu(\frac{n_2 R \nu t}{a})$ in $\Delta_\nu^{TM}(z)$.

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- [1] H.B.G. Casimir. Proc. Kon. Nederl. Akad. Wet., 51:793–796, 1948.
 - [2] V. M. Mostepanenko and N. N. Trunov. *The Casimir effect and its applications*. Oxford Science Publications, New York, 1997.
 - [3] M. Bordag, U. Mohideen, V.M. Mostepanenko. *New Developments in the Casimir Effect*. To appear in Physics Reports, quant-ph/0106045, 2001.
 - [4] J. Schwinger. *Proc. Nat. Acad. Sci.*, 89, (1992) 4091; 89, (1992) 11118; 90, (1993) 958; 90, (1993) 2105; 90, (1993) 4505; 90, (1993) 7285; 91, (1994) 6473.
 - [5] R. Löfstedt B.P. Barber, R.A. Hiller and S.J. Putterman. Physics Reports, 281:65–143, 1997.
 - [6] H.B.G. Casimir and D. Polder. Phys. Rev. **73**, 360 (1948).
 - [7] L.D. Landau and E.M. Lifshitz. *Electrodynamics of Continuous Media*. Pergamon, Oxford, (1960).
 - [8] J. Schwinger, L.L. DeRaad and K.A. Milton. Ann. Phys. **115**, 1 (1978).
 - [9] K.A. Milton. Ann. Phys. **127**, 49 (1980).
 - [10] P. Candelas. Ann. Phys. **143**, 241 (1982).
 - [11] J. Baake and G. Krüsermann. Z. Phys. C **30**, 413 (1986).
 - [12] G. Plunien, B. Müller and W. Greiner. Phys. Rep. **134**, 87 (1986).
 - [13] K.A. Milton. *The Casimir Effect: Physical Manifestations of Zero-Point Energy*. hep-th/9901011 (1999).
 - [14] I. Brevik, I. Skurdal and R. Solie. J. Phys. A: Math. Gen. **27**, 6853 (1994).
 - [15] I. Brevik and T. A. Yousef. J. Phys. A: Math. Gen. **33**, 5189 (2000).
 - [16] J. S. Høye and I. Brevik. Phys. Rev. E63:051101,2001.
 - [17] I. Brevik, V.N. Marachevsky and K.A. Milton. Phys. Rev. Lett. **82**, 3948 (1999).
 - [18] G. Barton. J. Phys. A: Math. Gen. **32**, 525 (1999).
 - [19] Claudia Eberlein. *Theory of quantum radiation observed as sonoluminescence*. quant-ph/9506024 (1995).
 - [20] Claudia Eberlein. *Sonoluminescence as quantum vacuum radiation*. quant-ph/9506023 (1995).
 - [21] Kimball A. Milton and Y. Jack Ng. *Casimir energy for a spherical cavity in a dielectric: Applications to sonoluminescence*. hep-th/9607186 (1996).
 - [22] C. Molina-Paris and Matt Visser. Phys. Rev. D56:6629–6639 (1997).
 - [23] C. E. Carlson, C. Molina-Paris, J. Perez-Mercader, and Matt Visser. Phys. Rev. D56:1262–1280 (1997).
 - [24] Kimball A. Milton and Y. Jack Ng. Phys. Rev. E57:5504 (1998).
 - [25] M. Bordag, K. Kirsten, and D. Vassilevich. Phys. Rev. D59:085011 (1999).
 - [26] G. Lambiase, G. Scarpetta, and V. V. Nesterenko. *Exact value of the vacuum electromagnetic energy of a dilute dielectric ball in the mode summation method*. hep-th/9912176 (1999).
 - [27] S. Liberati, Matt Visser, F. Belgiorno, and D. W. Sciama. Phys. Rev. D61:085023 (2000).

- [28] S. Liberati, Matt Visser, F. Belgiorno, and D. W. Sciama. Phys. Rev. D61:085024 (2000).
- [29] V.N. Marachevsky. *Casimir energy and dilute dielectric ball*. hep-th/0010214 (2000).
- [30] V.N. Marachevsky. *Casimir energy and realistic model of dilute dielectric ball*. hep-th/0101062 (2001).
- [31] H. Falomir, K. Kirsten and K. Rébora. *Divergencies in the Casimir energy for a medium with realistic ultraviolet behavior*. To appear in the Journal of Physics A: Mathematical and General, hep-th/0103050 (2001).
- [32] I. Brevik and V.N. Marachevsky. Phys. Rev. D60, 085006 (1999).
- [33] H. Falomir, M. Loewe and K. Rébora. Phys. Rev. D63, 025015 (2001).
- [34] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions*. Dover Publications, Inc., New York, 1972.
- [35] G. H. Hardy. *Divergent Series*. AMS Chelsea Publishing, Providence, Rhode Island, 1991.
- [36] J. D. Jackson. *Classical Electrodynamics, 2nd. ed.* Wiley, New York, 1975.
- [37] M. Reed and B. Simon. *Methods of modern mathematical analysis II: Fourier analysis and self-adjointness*. Wiley, New York, 1975.